On the Strategy-Proof and Smooth Allocation of Private Goods in Production Economies

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Abstract

We consider the problem of allocating private goods in a production economy, which is an economy where the total consumption bundle must lie on some smooth manifold in $\Re^m_+$. We describe a family of social choice functions called *generalized serial mechanisms* which have the extremely strong strategic property that (1) a utility profile is a Nash equilibrium if and only if the profile produces the truthful outcome, and (2) every Nash equilibrium is a strict strong equilibrium (i.e., at least one member in every deviating coalition that perturbs the outcome must be strictly less well off). We then introduce certain smoothness (differentiability) conditions and show that any anonymous, nonconstant, and smooth social choice function which is coalitionally strategy-proof, or is strategy-proof and nonbossy, must be a generalized serial mechanism. This work is an outgrowth of Moulin-Shenker (1991) which discusses the *serial* cost sharing method. This work also relies heavily on the result of Satterthwaite and Sonnenschein (1981) which states that a nonbossy and strategy-proof social choice function must be acyclic at every differentiable point.

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1 Introduction

We consider the smooth and strategy-proof allocation of private goods in production economies, which are economies where the total consumption bundle is constrained to lie on some smooth manifold. For these production economies, we describe a family of social choice functions called *generalized serial mechanisms* which have the unusually strong strategic property (Theorem 1) that (1) a utility profile is a Nash equilibrium if and only if the profile produces the truthful outcome, and (2) every Nash equilibrium is a strict strong equilibrium (i.e., at least one member in every deviating coalition that perturbs the outcome must be strictly less well off).

Satterthwaite and Sonnenschein (1981) showed that strategy-proof and nonbossy social choice functions must be acyclic (that is, locally a serial dictatorship) at every differentiable point. Our present paper is an extension of that line of investigation; we ask if the combination of smoothness, (coalitional) strategy-proofness, and nonbossiness not only implies the local property of acyclicity, but also leads to a global characterization. We find (Theorem 2) that any anonymous, nonconstant, and smooth social choice function (where the term *smooth* denotes a whole set of differentiability conditions and boundary conditions) which is either coalitionally strategy-proof, or strategy-proof and nonbossy, must be a generalized serial mechanism.

There is a large literature on strategy-proofness, and that literature has enjoyed a recent flurry of activity. In particular, there have been several recent papers examining strategy-proofness in economies with private goods. Barberà and Jackson (1992) characterize the set of strategy-proof social choice functions in exchange economies. This result is the first general characterization of strategy-proof mechanisms with private goods and, in contrast to what is presented in this paper, their results are unsullied by any smoothness conditions on the social choice functions. Unfortunately, this characterization result does not automatically extend to production economies, where the total bundle of goods is not fixed. Zhou (1991b) shows that in a two-person exchange economy, any efficient strategy-proof social choice function must be inversely dictatorial. Sprumont (1992) shows that any continuous and strategy-proof mechanism in a two-good, two-agent exchange economy must allow one agent to choose her best bundle from some exogenously determined set. These two results, as is the case with Barberà and Jackson's characterization result, do not directly extend to the production economies considered in this paper; furthermore, Zhou's and Sprumont's results also do not directly extend to the general n-agent case.

For economies with purely public goods, Barberà and Jackson (1991) characterize the set of strategy-proof social choice functions. This characterization builds on Moulin's (1980) characterization of strategy-proof mechanisms with one good and single-peaked preferences, and it also builds on Zhou's (1991a) result that any nondictatorial strategy-proof mechanism in this context must have a range of dimension one. Strategy-proofness in the voting context has been recently treated in Barberà-Peleg (1990) and in Barberà-Sonnenschein-Zhou (1991).

Thus, the class of strategy-proof mechanisms for public good economies and for private good exchange economies are fully characterized. However, there are few results applicable to more general private good economies. In particular, the Satterthwaite and Sonnenschein (1981) result is the only one of the aforementioned results directly applicable to production economies, and consequently we base our investigation on this result.

This paper has 6 sections. We provide the relevant notation and definitions in Section 2. Generalized serial mechanisms are defined and discussed in Section 3. The precise statement of the characterization theorem (Theorem 2) is presented in Section 4. Section 5 addresses the more accessible case where there are only two goods, presenting both the precise statement and the proof of the analogous characterization result (Theorem 3). Due to its length, we delay the detailed proof of Theorem 2, the general form of the characterization result, until Section 6.

2 Notation and Definitions

For any positive integer k, we use the notation $\iota(k) = \{1, 2, \ldots, k-1, k\}$. For any vector $\vec{q} \in \Re^k$ and any set of disjoint subsets $E_i \subseteq \iota(k)$, we use the notation $(\vec{q}|^{E_1;E_2;\cdots} z_1, z_2, \ldots)$ to denote the vector with $(\vec{q}|^{E_1;E_2;\cdots} z_1, z_2, \ldots)_i = q_i$ for all $i \notin \cup_j E_j$ and $(\vec{q}|^{E_1;E_2;\cdots} z_1, z_2, \ldots)_i = z_j$ for all $i \in E_j$. In particular, $(\vec{q}|^i z)$ denotes the vector with $(\vec{q}|^i z)_j = q_j$ for all $j \neq i$ and $(\vec{q}|^i z)_i = z$; similarly, for any subset $E \subseteq \iota(k)$ we define $(\vec{q}|^E z)$ to be the vector with $(\vec{q}|^E z)_i = q_i$ for all $i \notin E$ and $(\vec{q}|^E z)_i = z$ for all $i \in E$. For a set X, $\overline{X}$ denotes its closure and ∂X denotes its boundary.

We are considering a private goods problem with n agents and m goods, with $n > 1$ and $m > 1$; we use the notation that $N = \iota(n)$ and $M = \iota(m)$. Let $\mathcal{U}$ denote the set of functions $U : \Re^m_+ \mapsto \Re$ which are continuous, nondecreasing in each dimension, nonlocally satiated, and quasi-concave. The domain of preferences are those that are representable by members of $\mathcal{U}$. We use the standard topology on $\mathcal{U}$. Let $\mathcal{L}$ denote the linear subset of $\mathcal{U}$, which is isomorphic to $\Re^m_+ - \{0^m\}$. In this paper we use the functional representation of utilities rather than the more cumbersome binary relation representation; however, all of our definitions and our results are purely ordinal.

A production economy is one in which goods can be transformed into other goods. The production technology, that is the set of realizable total consumption bundles, is represented by some smooth $(m-p)$-dimensional manifold $\mathcal{P}$, with $m > p > 0$; let $P = \iota(p)$. This manifold is defined by the constraint $\vec{A}(x) = 0$ where $\vec{A} : \Re^m_+ \mapsto \Re^p$ is a twice continuously differentiable constraint function with $A^\nu(\vec{0}) < 0$ and $\frac{\partial A^\nu}{\partial x^\alpha} > 0$ for all $\alpha \in M$ and $\nu \in P$. We require that the matrix $\frac{\partial A^\nu}{\partial x^\alpha}$ have null space of minimal dimension $(m-p)$ for all $x \in \Re^m_+$. Let $\mathcal{A} \subseteq \Re^{mn}_+$ denote the set of feasible allocation vectors: $\mathcal{A} \equiv \{\vec{x} \in \Re^{mn}_+ | (\sum_{i=1}^n x_i) \in \mathcal{P}\}$.

In this paper, we consider the set $\mathcal{S}$ of direct allocation mechanisms, or social choice functions, $\vec{S} : \mathcal{U}^n \mapsto \mathcal{A}$. We will be focussing on the strategic properties of these social choice functions, and so we start with the standard definition of strategy-proofness: $\vec{S}$ is *strategy-proof* (SP) if, for all $\vec{U} \in \mathcal{U}^n$ and all $i \in N$, $U_i(S_i(\vec{U})) \geq U_i(S_i(\vec{U}|^i V_i))$ for all $V_i \in \mathcal{U}$. In our characterization results, we use strategic properties that are slightly stronger than strategy-proofness. One augmentation to strategy-proofness is the condition of *nonbossiness* (NB) which was introduced by Satterthwaite and Sonnenschein (1981): $\vec{S}$ is nonbossy if, for all $i \in N$, $\vec{U} \in \mathcal{U}^n$, and $V_i \in \mathcal{U}$, $S_i(\vec{U}) = S_i(\vec{U}|^i V_i) \Rightarrow \vec{S}(\vec{U}) = \vec{S}(\vec{U}|^i V_i)$. Another augmentation to strategy-proofness is to consider coalitional behavior: S is *coalitionally strategy-proof* (CSP) if, for all $\vec{U}, \vec{V} \in \mathcal{U}^n$, either $U_j(S_j(\vec{U})) = U_j(S_j(\vec{V}))$ for all j such that $U_j \neq V_j$, or there exists some j such that $U_j \neq V_j$ and $U_j(S_j(\vec{V})) < U_j(S_j(\vec{U}))$. We will consider, in our characterization results, the set of social choice functions which are either coalitionally strategy-proof, or strategy-proof and nonbossy.

Obviously the constant social choice function, where the allocation $\vec{S}(\vec{U})$ is independent of $\vec{U}$, is coalitionally strategy-proof and nonbossy. The first question we address is: are there any nonconstant social choice functions in production economies which are either coalitionally strategy-proof, or strategy-proof and nonbossy? The answer is that there are indeed such social choice functions, at least when the manifold $\mathcal{P}$ admits certain curves (defined by a generalized serial function to be defined below) to be embedded within it. We now turn, in the next section, to the description of a family of such mechanisms.

3 Generalized Serial Mechanisms

In Moulin-Shenker (1991), we introduced a social choice function called the *serial* mechanism for a cost-sharing problem in which there is a single good and a single bad (the cost), with the total bad being a function of the total good. The nomenclature reflects that fact that this mechanism is everywhere a local serial dictatorship. Mechanisms similar to the serial mechanism can easily be defined for an $m = 2$, $p = 1$ version of the production economy we are considering in this paper; we discuss this case more fully in Section 5. In this section, we define a more general set of mechanisms which extends the intuition behind the original serial mechanism to arbitrary values of m and p. We first define a set of functions called *generalized serial functions*, and then use these to define the set of *generalized serial mechanisms.*

Consider a function $\vec{a} : [0,1]^n \mapsto \Re_+^{nm}$ such that, for all $\vec{z} \in [0,1]^n$ and all $\lambda \in \mathcal{L}$: (1) $z_i \leq z_j \Rightarrow a_i(\vec{z}) = a_i(\vec{z}|^j s)$ for all $s \in [z_i, 1]$, (2) $\lambda \cdot a_i(\vec{z}|^i s)$ has a unique maximizer $\bar{s}_i$ for each i, (3) if $\bar{s}_i$ is the unique maximizer of $\lambda \cdot a_i(\vec{z}|^i s)$ then $\bar{s}_i$ is also the unique maximizer of $\lambda \cdot a_i(\vec{z}'|^i s)$ for all $\vec{z}'$ such that, for all $j \neq i$, $MIN[z'_j, z_j] < \bar{s}_i \Rightarrow z'_j = z_j$, and (4) $\sum_{i=1}^n a_i(\vec{z}) \in \mathcal{P}$. The second requirement is a convexity condition and the fourth requirement is a feasibility condition. The first requirement is a seriality condition; i.e., the function $\vec{a}(\vec{z})$ is locally a serial dictatorship in that a sufficiently small change to z_j does not change $a_i(\vec{z})$ if $z_j > z_i$. Similarly, the third requirement is a seriality condition on the maximizers $\bar{s}_i$ in that a sufficiently small change to z_j does not change $\bar{s}_i$ if $z_j > z_i$. Because of the seriality embedded in these requirements, we will call such a function $\vec{a}(\vec{z})$ a *generalized serial function*, and let $\mathcal{F}$ denote the set of such functions. Associated with each generalized serial function $\vec{a} \in \mathcal{F}$ and each utility profile $\vec{U} \in \mathcal{U}$ is a normal form game where the strategy spaces are the intervals $[0,1]$ and the payoff to each agent i resulting from a given strategy vector $\vec{z}$ is given by $U_i(a_i(\vec{z}))$. Such normal forms games have the following relevant property.

Lemma 1 *For any $\vec{U} \in \mathcal{U}^n$ and any $\vec{a}(\vec{z}) \in \mathcal{F}$, the associated normal form game has a unique Nash equilibrium.*

Proof: Consider some $\vec{U} \in \mathcal{U}^n$ and $\vec{a}(\vec{z}) \in \mathcal{F}$. First we show that there cannot be more than one Nash equilibrium, and then we will show that there must be at least one Nash equilibrium. Before proceeding, we first note that because the utilities U_i are convex the second and third conditions in the definition of a generalized serial function imply that, for each i, $U_i(a_i(\vec{z}|^i s))$ has a unique maximizer $\bar{s}_i$, and that $\bar{s}_i$ is also the unique maximizer of $U_i(a_i(\vec{z}'|^i s))$ for all $\vec{z}'$ such that, for all $j \neq i$, $MIN[z'_j, z_j] < \bar{s}_i \Rightarrow z'_j = z_j$.

Assume, to the contrary, that there are two distinct Nash equilibria $\vec{z}$ and $\vec{z}'$. We know that for each agent j, z_j is the unique maximizer of $U_j(a_j(\vec{z}|^j s))$ and z'_j is the unique maximizer of $U_j(a_j(\vec{z}'|^j s))$. Let i be an element such that $z_i \neq z'_i$ and $MIN[z_j, z'_j] < MIN[z_i, z'_i] \Rightarrow z_j = z'_j$. Assume, without loss of generality, that $z'_i < z_i$. Then, both z_i and z'_i are maximizers of $U_i(a_i(\vec{z}|^i s))$ which is impossible given the unique maximizer condition in the definition of generalized serial functions. Thus, there can be no more than one Nash equilibrium.

To show that there must be at least one Nash equilibrium, we now provide an explicit construction of a Nash equilibrium. Set $\vec{z} = (1, 1, \ldots, 1, 1)$. For each i, define s_i^1 as the unique value of s that maximizes $U_i(a_i(\vec{z}|^i s))$. Assume, without loss of generality, that s_1^1 is the minimal such s_i^1, and set $z_1 = s_1^1$ (leaving the other elements of $\vec{z}$ unchanged). Define, for each i, s_i^2 as the unique value of s that maximizes $U_i(a_i(\vec{z}|^i s))$. Note that $s_1^2 = z_1$, since it is the same optimization problem that led to s_1^1, and $s_i^2 \geq z_1$ for $i \neq 1$, since $s_i^2 < z_1 \Rightarrow s_i^1 = s_i^2 < z_1$ which cannot hold by assumption for $i > 1$. Assume, without loss of generality, that s_2^2 is the minimal such s_i^2 with $i > 1$, and set $z_2 = s_2^2$ (leaving the other elements of $\vec{z}$ unchanged). Define, for each i, s_i^3 as the unique value of s that maximizes $U_i(a_i(\vec{z}|^i s))$. Note that $s_1^3 = z_1$ and $s_2^3 = z_2$, since these must, as per condition (3) in the definition of a generalized serial function, still be the unique maximizing values. Furthermore, $s_i^3 \geq z_2$ for $i > 2$, since $s_i^3 < z_2 \Rightarrow s_i^2 = s_i^3 < z_2$ which cannot hold by assumption for $i > 2$. Assume, without loss of generality, that s_3^3 is the minimal such s_i^3 with $i > 2$, and set $z_3 = s_3^3$ (leaving the other elements of $\vec{z}$ unchanged). Repeat this process until $\vec{z}$ is fully redefined. This will have produced a vector $\vec{z} \in [0,1]^n$ with $z_i \leq z_{i+1}$. Furthermore, for each i, z_i is the unique maximizer of $U_i(a_i(\vec{z}|^i s))$, so this construction produces the unique Nash equilibrium of the associated normal form game. □

Given any generalized serial function $\vec{a}(\vec{z}) \in \mathcal{F}$, we can define a social choice function $\vec{S}$, called a *generalized serial mechanism*, as follows. For each $\vec{U} \in \mathcal{U}^n$, define $\vec{S}(\vec{U})$ to be the allocations $\vec{a}(\vec{z})$ resulting from the unique Nash equilibrium $\vec{z}$ of the associated normal form game. These generalized serial mechanisms are coalitionally strategy-proof and nonbossy; in fact, they have a much stronger nonmanipulability property (which is similar to the property called *full incentive compatibility* in Shenker (1992)).

Theorem 1 *If $\vec{S}$ is a generalized serial mechanism then, for any $\vec{U}, \vec{W} \in \mathcal{U}^n$, (1) $U_i(S_i(\vec{W})) \geq U_i(S_i(\vec{W}|^i V))$ for all V and all i if and only if $\vec{S}(\vec{U}) = \vec{S}(\vec{W})$, and (2) if $\vec{S}(\vec{U}) = \vec{S}(\vec{W})$ then there is no $\vec{V}$ such that $\vec{S}(\vec{V}) \neq \vec{S}(\vec{U})$ and $U_i(S_i(\vec{U})) \leq U_i(S_i(\vec{V}))$ for all i with $W_i \neq V_i$.*

Proof: Consider a generalized serial function $\vec{a}(\vec{z}) \in \mathcal{F}$ and the resulting generalized serial mechanism $\vec{S}$. Consider any profile $\vec{U}$ and the Nash equilibrium $\vec{z}$ of the associated normal form game: $\vec{a}(\vec{z}) = \vec{S}(\vec{U})$. We know that, for all i, $U_i(a_i(\vec{z}|^i s)) < U_i(a_i(\vec{z}))$ for all $s \neq z_i$.

For two vectors $\vec{x}, \vec{y} \in [0,1]^n$, let us denote by $MD(\vec{x}, \vec{y})$ the set of elements k such that $x_k \neq y_k$ and $x_i = y_i$ for all i such that $MIN[x_i, y_i] < MIN[x_k, y_k]$. If $\vec{x} \neq \vec{y}$, this set $MD(\vec{x}, \vec{y})$ will have either one or two members. We claim the following: for any $\vec{z}' \in [0,1]^n$, $\vec{a}(\vec{z}) = \vec{a}(\vec{z}') \Rightarrow \vec{z} = \vec{z}'$. This follows because if $\vec{a}(\vec{z}) = \vec{a}(\vec{z}')$ but $\vec{z} \neq \vec{z}'$ then for any $k \in MD(\vec{z}, \vec{z}')$ either $U_k(a_k(\vec{z}|^k s))$ or $U_k(a_k(\vec{z}'|^k s))$ (depending on whether $z_k > z'_k$ or $z_k < z'_k$) is maximized at both z_k and z'_k.

Consider any profile $\vec{W}$ such that $\vec{S}(\vec{W}) = \vec{S}(\vec{U})$, and thus also has $\vec{z}$ as the Nash equilibrium of the associated normal form game. Consider any profile $\vec{V}$ such that $\vec{S}(\vec{V}) \neq \vec{S}(\vec{U})$ and let $\vec{t}$ denote the Nash equilibrium of the associated normal form game. Choose some $k \in MD(\vec{z}, \vec{t})$. Let $\vec{r}$ denote the Nash equilibrium of the normal form game associated with the profile $(\vec{V}|^k U_k)$. There are two cases. If $z_k < t_k$ then $r_k = z_k$ and $k \in MD(\vec{r}, \vec{t})$, so $a_k(\vec{z}) = a_k(\vec{r})$. By construction, t_k is the maximizer of $V_k(a_k(\vec{t}|^k s))$; however, t_k is not the maximizer of $U_k(a_k(\vec{t}|^k s))$, nor is t_k the maximizer of $W_k(a_k(\vec{t}|^k s))$, because otherwise we would have $z_k = t_k$. Thus, $V_k \neq U_k$, $V_k \neq W_k$, and $U_k(a_k(\vec{z})) = U_k(a_k(\vec{r})) = U_k(a_k(\vec{t}|^k r_k)) > U_k(a_k(\vec{t}))$. If $z_k > t_k$ then $r_k > t_k$ and $k \in MD(\vec{r}, \vec{t})$. Furthermore, $U_k(a_k(\vec{r})) > U_k(a_k(\vec{r}|^k t_k)) = U_k(a_k(\vec{t}))$ and $U_k(a_k(\vec{z})) > U_k(a_k(\vec{z}|^k t_k)) = U_k(a_k(\vec{t}))$; also, by the same reasoning as in the previous case, $V_k \neq U_k$ and $V_k \neq W_k$. Therefore, in both cases, $U_k(S_k(\vec{V}|^k U_k)) > U_k(S_k(\vec{V}))$, $U_k(S_k(\vec{U})) > U_k(S_k(\vec{V}))$, $V_k \neq W_k$, and $V_k \neq U_k$.

We have shown that for any profiles $\vec{U}, \vec{W}, \vec{V}$ such that $\vec{S}(\vec{W}) = \vec{S}(\vec{U}) \neq \vec{S}(\vec{V})$ there must exist some element k such that $V_k \neq W_k$, $U_k(S_k(\vec{V}|^k U_k)) > U_k(S_k(\vec{V}))$, and $U_k(S_k(\vec{U})) > U_k(S_k(\vec{V}))$. This establishes condition (2) of the theorem.

Consider profiles $\vec{V}$ of the form $\vec{V} = (\vec{W}|^i V)$. Then, the above result shows that for any $\vec{W}$, $U_i(S_i(\vec{W})) \geq U_i(S_i(\vec{W}|^i V))$ for all V and all i if and only if we have $\vec{S}(\vec{U}) = \vec{S}(\vec{W})$. This establishes condition (1) of the theorem. $\square$

Our results so far in this section have delineated the strategic properties of generalized serial mechanisms. However, this leaves unanswered the question of whether a given production technology can support any generalized serial mechanisms at all; in short, for which manifolds $\mathcal{P}$ is the set $\mathcal{F}$ empty? We do not provide an answer to this question here. However, we will now describe two conditions, a necessary condition and a sufficient condition, for the set $\mathcal{F}$ to be nonempty. Both conditions pivot on the possibility of embedding certain curves in $\mathcal{P}$; we leave for more informed researchers the question of which manifolds support such embeddings.

Consider some production technology $\mathcal{P}$ and assume that there exists at least one generalized serial function $\vec{a}(\vec{z})$. Consider the curve $b(s) = na_1(\vec{z}|^N s)$ (which is clearly independent of $\vec{z}$). For each $s \in [0,1]$, $b(s) \in \mathcal{P}$. Furthermore, $\lambda \cdot b(s)$ always has a unique maximizer $\bar{s}$ for all $\lambda \in \mathcal{L}$ because $b(s) = na_1(\vec{z}|^1 s)$ when $\vec{z} = (1, \ldots, 1)$. Thus, a necessary condition for a production technology to allow the embedding of a generalized serial function is that there must be some curve $b : [0,1] \mapsto \mathcal{P}$ such that $\lambda \cdot b(s)$ always has a unique maximizer for all $\lambda \in \mathcal{L}$. Not all manifolds possess this property; consider, for instance, the manifold $x^1 + x^2 = 1$ in $\Re^2_+$. Since every point yields the same value for $\lambda \cdot x$ when $\lambda = (1,1)$, the dot product $(1,1) \cdot a_1(\vec{z}|^N s)$ does not have a unique maximizer. Therefore, $\mathcal{F}$ is empty for this manifold.

The aforementioned necessary condition for $\mathcal{F}$ to be nonempty involves the existence of a curve that satisfied a weak convexity property. We now present a sufficient condition for $\mathcal{F}$ to be nonempty which involves the existence of a curve that satisfies a significantly stronger convexity property. The set $\mathcal{F}$ is nonempty if $\mathcal{P}$ contains some C^2 curve $b : [0,1] \mapsto \mathcal{P}$ such that $b(s) \in \Re^m_{++}$ and $b''(s) \in \Re^m_{--}$ for all $s \in (0,1)$. To show that this is indeed a sufficient condition, we must construct a generalized serial function $\vec{a}(\vec{z})$ assuming only the existence of such a curve $b(s)$. For each $\vec{z} \in (0,1)^n$ with $z_j < z_{j+1}$ for all j, define $a_i(\vec{z})$ recursively as follows. First, set $a_1(\vec{z}) = \frac{b(z_1)}{n}$ and $r_1 = z_1$. Second, given $a_i(\vec{z}) \in \Re^m_{++}$ and $r_i \in (0,1)$ for all $i \leq j$, consider the curve $f_j(s) = a_j(\vec{z}) + \frac{b(\frac{(n-j)s + r_j + \ldots + r_1}{n}) - b(\frac{(n-j+1)r_j + r_{j-1} + \ldots + r_1}{n})}{n-j}$. Note that $f_j(r_j) = a_j(\vec{z})$ and that if $a_j(\vec{z}) \in \Re^m_{++}$ then, for small enough ϵ, $f_j(r_j+\epsilon) \in \Re^m_{++}$. If $f_j(s) \in \Re^m_{++}$ for all $r_j < s \leq 1$ then set $\bar{s}_j = 1$; otherwise, let $\bar{s}_j$ denote the minimal value of $s > r_j$ such that $f_j(s) \notin \Re^m_{++}$. Let $r_{j+1} = r_j + \frac{(z_{j+1}-z_j)(\bar{s}_j - r_j)}{1-z_j}$ and then set $a_{j+1}(\vec{z}) = f_j(r_{j+1})$; note that $a_{j+1}(\vec{z}) \in \Re^m_{++}$. If we then impose anonymity (i.e., symmetry) and continuity, then $\vec{a}(\vec{z})$ is defined for all $\vec{z} \in [0,1]^n$. Conditions (1) and (4) in the definition of a generalized serial function are satisfied trivially. Since $b(s)$ is C^2 and $b''(s) \in \Re^m_{--}$ for all $s \in (0,1)$ we know that $\lambda \cdot b(s)$ has a unique maximizer for all $\lambda \in \mathcal{L}$ and that $\bar{s} \in (0,1)$ is the maximizer if and only if $\lambda \cdot b'(\bar{s}) = 0$. While the function $\vec{a}(\vec{z})$ defined above is not C^1, it is true that for all i and all $s \neq z_j$ for all j, $a_i(\vec{z}|^i s)$ is locally C^2 and has $\frac{\partial^2 a_i}{\partial s^2}(\vec{z}|^i s) \in \Re^m_{--}$. At the points where $s = z_j$, the onesided derivatives exist and are parallel to each other. Therefore, we can conclude that $\lambda \cdot a_i(\vec{z}|^i s)$ has a unique maximizer for all $\lambda \in \mathcal{L}$ and that $\bar{s} \in (0,1)$ is the maximizer if and only if $\lambda \cdot \frac{\partial a_i}{\partial s^+}(\vec{z}|^i s) = \lambda \cdot \frac{\partial a_i}{\partial s^-}(\vec{z}|^i s) = 0$ at that point. Consequently, conditions (2) and (3) in the definition of a generalized serial function are satisfied. Therefore, this function $\vec{a}(\vec{z})$ is a generalized serial function and so $\mathcal{F}$ is nonempty. This establishes that the aforementioned condition is indeed sufficient to guarantee the nonemptiness of $\mathcal{F}$. We will call this construction the *serializing* construction of a function $b(s)$.

This serializing construction produces a generalized serial mechanism which is very similar to the original serial mechanism discussed in Moulin-Shenker (1991). As we shall see in Section 5, the serializing construction produces the only possible generalized serial mechanisms when there are only two goods ($m = 2$). When we have only linear utilities, with $U_i(x) = \lambda_i \cdot x$, the generalized serial mechanism that results from such a serializing construction has a much simpler representation. Denote by t_i the unique maximizing value of $\lambda_i \cdot b(s)$. Assume that $t_1 \leq t_2 \leq \ldots t_{n-1} \leq t_n$. Then, $S_1(\vec{U}) = \frac{b(t_1)}{n}$ and, for $1 < i \leq n$, $S_i(\vec{U}) = S_{i-1}(\vec{U}) + \frac{b(MIN[t_i,\bar{t}_i]) - b(MIN[t_{i-1},\bar{t}_{i-1}])}{n-i+1}$, where $\bar{t}_1 = 0$ and $\bar{t}_i$ for $i > 1$ is defined to be the smallest value of $s \in [\bar{t}_{i-1}, 1]$ such that $S_{i-1}(\vec{U}) + \frac{b(s) - b(MIN[t_{i-1},\bar{t}_{i-1}])}{n-i+1} \notin \Re^m_{++}$. This representation makes it obvious that the resulting generalized serial mechanism is independent of the parameterization of $b(s)$ (as long as it is invertible and retains the same orientation as the original parameterization).

4 Characterization Theorem

The generalized serial mechanisms described in the previous section provide examples of non-constant and anonymous social choice functions which are both coalitionally strategy-proof and nonbossy. In this section we address the issue of whether there are any other social choice functions which are either coalitionally strategy-proof or strategy-proof and nonbossy. It will turn out that if we also require that these social choice functions be smooth then there are no other such social choice functions. In this section, we first describe the details of the smoothness condition and then state our main characterization result. We start by describing the implications of requiring coalitionally strategy-proof or strategy-proof and nonbossy social choice functions to be continuous; the implications of continuity for such social choice functions allows for the significant technical simplification of considering only linear preferences.

4.1 Implications of Continuity

We say that $\vec{S}$ is anonymous if $\vec{S}$ is invariant under permutations σ: $\sigma(\vec{S}(\vec{U})) = \vec{S}(\sigma(\vec{U}))$. We say that $\vec{S}$ is weakly anonymous if $S_i(\vec{U}) = S_j(\vec{U})$ whenever $U_i = U_j$. Let $\tilde{\mathcal{S}}$ denote the set of functions $\vec{S} \in \mathcal{S}$ that are continuous, weakly anonymous, and either coalitionally strategy-proof, or strategy-proof and nonbossy. In Shenker (1992), we show that every $\vec{S} \in \tilde{\mathcal{S}}$ has the following properties.

Lemma 2 *If $\vec{S} \in \tilde{\mathcal{S}}$ then:*

1. *$\vec{S}$ is strictly strategy-proof (SSP): for any $\vec{U} \in \mathcal{U}^n$ and $V_i \in \mathcal{U}$, $S_i(\vec{U}) \neq S_i(\vec{U}|^i V_i) \Rightarrow U_i(S_i(\vec{U})) > U_i(S_i(\vec{U}|^i V_i))$.*
2. *$\vec{S}$ is weakly coalitionally strategy-proof (WCSP): For any pair $\vec{U} \neq \vec{V} \in \mathcal{U}^n$ there exists some j such that $U_j \neq V_j$ and $U_j(S_j(\vec{V})) \leq U_j(S_j(\vec{U}))$.*
3. *$\vec{S}$ is coalitionally nonbossy (CNB): for any pair $\vec{U}, \vec{V} \in \mathcal{U}^n$, $\vec{S}(\vec{U}) = \vec{S}(\vec{V})$ whenever $S_i(\vec{U}) = S_i(\vec{V})$ for each i such that $U_i \neq V_i$.*
4. *$\vec{S}$ is Maskin monotonic (see Maskin (1985)): $\vec{S}(\vec{V}) = \vec{S}(\vec{U})$ whenever $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow V_i(x) \leq V_i(S_i(\vec{U}))$ for all allocations x and all i.*
5. *The allocations are envy-free but not mutually indifferent: for all i, j and all $\vec{U} \in \mathcal{U}^n$ with $S_i(\vec{U}) \neq S_j(\vec{U})$, $U_i(S_i(\vec{U})) \geq U_i(S_j(\vec{U}))$ and $U_i(S_i(\vec{U})) = U_i(S_j(\vec{U})) \Rightarrow U_j(S_i(\vec{U})) < U_j(S_j(\vec{U}))$.*
6. *$\vec{S}$ is uniquely characterized by its behavior on $\mathcal{L}^n$: If $\vec{S}, \vec{S}' \in \tilde{\mathcal{S}}$ have the property that $\vec{S}(\vec{L}) = \vec{S}'(\vec{L})$ for each $\vec{L} \in \mathcal{L}^n$, then $\vec{S}(\vec{U}) = \vec{S}'(\vec{U})$ for each $\vec{U} \in \mathcal{U}^n$.*
7. *$\vec{S}$ is fully described by its behavior on $\mathcal{L}^n$: for each $\vec{U} \in \mathcal{U}^n$, there exists some $\vec{L} \in \mathcal{L}^n$ such that $\vec{S}(\vec{U}) = \vec{S}(\vec{L})$.*

In what follows we restrict our attention (unless otherwise noted) to social choice functions $\vec{S} \in \tilde{\mathcal{S}}$. Given the results in the last two bullets in Lemma 2, we need only consider linear preferences. Thus, with a slight change in notation, we can consider $\vec{S}$ to be a mapping from $(\Re^m_+ - 0^m)^n$ to $\Re^{nm}_+$ where $\vec{S}(\vec{\lambda})$ denotes $\vec{S}(\vec{L})$ with $L_i(x) = \lambda_i \cdot x$.

4.2 Smoothness Conditions

We say that $\vec{\lambda} \in \Re^{mn}_{++}$ is a differentiable point of $\vec{S}$ if there is an open neighborhood $\mathcal{N} \subseteq \Re^{mn}_{++}$ of $\vec{\lambda}$ such that $\vec{S}$ is twice continuously differentiable on $\mathcal{N}$. We say a point $\vec{\lambda}$ is *smooth* if it is a differentiable point and, for each $l_i \in \Re^m$, the function $g(\epsilon) = S_i(\vec{\lambda}|^i \lambda_i + \epsilon l_i)$ either has $l_i \cdot g'(0) > 0$ or is constant (where defined). Define the *interior* linear utility profiles to be those $\vec{\lambda} \in \Re^{mn}_{++}$ that produce nonzero allocations:

$$\mathcal{X}(\vec{S}) \equiv \{\vec{\lambda} \in \Re^{mn}_{++} | \vec{S}(\vec{\lambda}) \in \Re^{mn}_{++}\}$$

At any profile $\vec{\lambda}$, we say that elements i, j can *coalesce* (or *fully coalesce*) if $S_j(\vec{\lambda}) = S_j(\vec{\lambda}|^j\lambda_i) = S_i(\vec{\lambda}|^i\lambda_j) = S_i(\vec{\lambda})$. We denote this reflexive and symmetric relation by $i \approx j$, and its negation by $i \not\approx j$. Define the set $\mathcal{Z}(\vec{S})$ as follows:

$$\mathcal{Z}(\vec{S}) = \{\vec{\lambda} \in \mathcal{X}(\vec{S}) | i \not\approx j \ \forall \ i \neq j\}$$

Clearly $\mathcal{Z}(\vec{S})$ is an open set in $\mathcal{X}(\vec{S})$ for all continuous $\vec{S}$.

We say a social choice function $\vec{S}$ is *Z-smooth* if it has the following four properties:

1. The set $\mathcal{X}(\vec{S})$ is nonempty, with $\vec{S}(\mathcal{L}^n) = \overline{\vec{S}(\mathcal{X}(\vec{S}))}$.
2. The second derivatives $\frac{\partial^2 S_k^\gamma}{\partial\lambda_i^\alpha \partial\lambda_j^\beta}(\vec{\lambda})$ are bounded on $\mathcal{Z}(\vec{S})$.
3. All $\vec{\lambda} \in \mathcal{Z}(\vec{S})$ are smooth points of $\vec{S}$.
4. If $\vec{\lambda} \notin \mathcal{X}(\vec{S})$, $l_i \in \Re^m$, and $\delta > 0$ are such that $(\vec{\lambda}|^i\lambda_i - \epsilon l_i) \in \mathcal{X}(\vec{S})$ for all $0 < \epsilon < \delta$, then $\vec{S}(\vec{\lambda}|^i\lambda_i + \epsilon l_i) = \vec{S}(\vec{\lambda})$ for all $\epsilon > 0$ such that $\lambda_i + \epsilon l_i \in \Re^m_+$.

Condition 4 requires that when changing a utility λ_i along some linear path, if the allocation $\vec{S}(\vec{\lambda})$ hits the boundary $\partial\Re^{mn}_+$ then the allocation $\vec{S}(\vec{\lambda})$ must remain constant as the profile continues to vary along the same linear path. This condition, while clearly unnatural, is necessary for our proof since we have absolutely no derivative conditions on the profiles outside of $\mathcal{X}(\vec{S})$.

4.3 Reduced Forms

Consider a partition $\{E_I\}$ of N: i.e. any complete set of disjoint subsets $E_I \subseteq N$, $I = 1, \ldots, K$ with $\cup_{I=1}^K E_I = N$ and $E_J \cap E_I = \emptyset$ for $I \neq J$. We can define the reduced form of $\vec{S}$, denoted by $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}$, corresponding to having the agents in each subset or *clique* E_I all signal the same utility U. Formally, $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K} : \mathcal{U}^{E_1;E_2;\ldots;E_{K-1};E_K} \mapsto \mathcal{A}$ with $\mathcal{U}^{E_1;E_2;\ldots;E_{K-1};E_K} \equiv \{\vec{U} \in \mathcal{U}^n | i, j \in E_I \ for \ some \ I \ \Rightarrow U_i = U_j\}$. When we restrict to linear profiles, we need to refer to the set $\mathcal{R}^{E_1;E_2;\ldots;E_{K-1};E_K} \equiv \{\vec{\lambda} \in \Re^{mn}_{++} | i, j \in E_I \ for \ some \ I \ \Rightarrow \lambda_i = \lambda_j\}$. One can think of this reduced form as representing an allocation among K agents, with a constraint function that is altered by requiring that, with the notation that i_J represents any element in the clique E_J, $(\sum_{J=1}^K |E_J| S_{i_J}(\vec{U})) \in \mathcal{P}$ for all $\vec{U} \in \mathcal{U}^{E_1;E_2;\ldots;E_{K-1};E_K}$. Let $\tilde{\mathcal{S}}^{E_1;E_2;\ldots;E_{K-1};E_K}$ denote the set of all functions $\vec{S} : \mathcal{U}^{E_1;E_2;\ldots;E_{K-1};E_K} \mapsto \mathcal{A}$ that, as functions of the U_{i_J}, are continuous, weakly anonymous, and either coalitionally strategy-proof or strategy-proof and nonbossy. If $\vec{S} \in \tilde{\mathcal{S}}$, then $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K} \in \tilde{\mathcal{S}}^{E_1;E_2;\ldots;E_{K-1};E_K}$.

The sets $\mathcal{X}(\vec{S})$ has an obvious counterpart for reduced forms:

$$\mathcal{X}(\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}) \equiv \{\vec{\lambda} \in \mathcal{R}^{E_1;E_2;\ldots;E_{K-1};E_K} | \vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}(\vec{\lambda}) \in \Re^{mn}_{++}\}$$

The generalization of the set $\mathcal{Z}(\vec{S})$ is also straightforward, if one generalizes the definition of coalescence to cliques:

$$\mathcal{Z}(\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}) = \{\vec{\lambda} \in \mathcal{X}(\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}) | E_i \not\approx E_j \ \forall \ i \neq j\}$$

where we say that cliques E_i, E_j can coalesce, $E_i \approx E_j$, if $S_{E_j}^{E_1;E_2;\ldots;E_{K-1};E_K}(\vec{\lambda}) = S_{E_j}^{E_1;E_2;\ldots;E_{K-1};E_K}(\vec{\lambda}|^{E_j}\lambda_{E_i}) = S_{E_i}^{E_1;E_2;\ldots;E_{K-1};E_K}(\vec{\lambda}|^{E_i}\lambda_{E_j}) = S_{E_i}^{E_1;E_2;\ldots;E_{K-1};E_K}(\vec{\lambda})$. We can then extend the definition of Z-smoothness of $\vec{S}$ to the reduced forms $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}$, by considering the reduced forms as functions from $\mathcal{U}^K$ to $\Re_+^{mK}$ or more particularly from $(\Re_+^m - 0^m)^K$ to $\Re_+^{mK}$. In particular, we say that a social choice function $\vec{S}$ is *recursively Z-smooth* if $\vec{S}$ is *Z-smooth* and each of its reduced forms $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}$, seen as functions from $(\Re_+^m - 0^m)^K$ to $\Re_+^{mK}$ on the linear profiles, are *Z-smooth.*

4.4 Statement of Characterization Theorem

With our smoothness conditions defined, we can now state our main characterization result.

Theorem 2 *Every recursively Z-smooth, nonconstant, and anonymous $\vec{S} \in \tilde{\mathcal{S}}$ is a generalized serial mechanism.*

We delay the proof of this theorem, and of the corollaries below, until Section 6. This theorem gives rise to several related results. First, the proof of the preceding theorem applies equally well if some of the quantities x^α are *bads* rather than *goods.* This allows us to apply our results to cost-sharing problems, which were the original motivation for the serial mechanism in Moulin-Shenker (1991). Second, we can define $\mathcal{X}(\vec{S})$ to be $\mathcal{X}(\vec{S}) \equiv \{\vec{\lambda} \in \Re_{++}^{mn} | \vec{S}(\vec{\lambda}) \in X\}$ for any open set $X \subseteq \Re_{++}^{mn}$ without changing the results of the theorem. Third, since every Nash implementable $\vec{S}$ must be strategy-proof and nonbossy (Shenker (1992)), the characterization result applies as well to Nash implementable social choice functions.

Corollary 1 *Any anonymous, continuous, nonconstant, Nash implementable, and recursively Z-smooth $\vec{S} \in \mathcal{S}$ must be a generalized serial mechanism.*

In the previous section we briefly discussed the question of which production technologies admit generalized serial mechanisms. The following is a relevant partial result. We say that a production technology has *increasing returns to scale* (IRS) if at every point on the manifold the matrices $\frac{\partial^2 A^\nu}{\partial x_\alpha \partial x_\beta}$ have non-negative eigenvalues.

Corollary 2 *If $\mathcal{P}$ has increasing returns to scale then there is no recursively Z-smooth, nonconstant, and anonymous $\vec{S} \in \tilde{\mathcal{S}}$.*

Satterthwaite and Sonnenschein (1981), in discussing their result that any smooth, strategy-proof, and nonbossy social choice function must be locally a serial dictatorship implicitly suggest that one might be able to achieve anonymity and continuity by piecing together such serial dictatorships. The serial mechanism, proposed in Moulin-Shenker (1991), is the first such example of a strategy-proof, nonbossy, and smooth mechanism which is also anonymous; the generalized serial mechanisms discussed here are further examples. However, the generalized serial mechanisms are not smooth everywhere. The following result states that this mutual exclusivity between smoothness everywhere and anonymity is necessary.

Corollary 3 *There is no anonymous, nonconstant, and recursively Z-smooth $\vec{S} \in \tilde{\mathcal{S}}$ for which every point in $\mathcal{X}(\vec{S})$ is a differentiable point.*

This result also explains, in part, why the smoothness conditions we invoked are rather bizarre; the generalized serial mechanisms are the only possible candidates, and they themselves have rather bizarre smoothness properties. In fact, the generalized serial mechanisms that result from the serializing construction discussed in Section 10 are not even C^1 at the points where elements coalesce (whereas the above corollary only precludes the existence of a C^2 mechanism). In addition, one can ask what happens when we remove the smoothness conditions altogether. Unfortunately, the present proof technique relies extremely heavily on the differentiability assumptions and so our results may not provide an accurate guide to this more general question. Thus, the challenging open problem is to characterize the set of anonymous and coalitionally strategy-proof (or strategy-proof and nonbossy) mechanisms in production economies.

5 Two Good ($m = 2$) Case

The case where there are only two goods deserves special attention. There are many examples of two-good production economies, with the single-input single-output cost sharing models examined in Moulin-Shenker (1991) being but one example. The two-good case also provides a more readily understandable illustration of the basic ideas behind the general m-good results. Because of the decreased dimensionality of the space, the proof of the theorem can be greatly simplified and the smoothness conditions can be relaxed slightly. With $m = 2$ we must have $p = 1$, since we had previously assumed $m > p > 0$. Thus, the production technology manifold $\mathcal{P}$ can be parametrized by a single variable, so there exists some invertible and C^2 function $b : [0,1] \mapsto \Re^m_+$ such that $b^1(0) = 0$, $b^2(1) = 0$, and $b([0,1]) = \mathcal{P}$, with $b'(s) \neq 0$, $b^1(s)$ strictly increasing, and $b^2(s)$ strictly decreasing. It will turn out that if there is any smooth and anonymous member of $\tilde{\mathcal{S}}$, then we can choose the parameterization of $b(s)$ such that $b''(s) \in \Re^m_{--}$. Then, we can define a mechanism, call it $\vec{S}^+$, by applying the serializing construction described in Section 3 to this function $b(s)$. Furthermore, we can define another mechanism, call it $\vec{S}^-$, by applying the serializing construction to the function $b(1-s)$. The characterization result in the $m = 2$ case is somewhat more specific than in the general m case, in that every smooth and anonymous member of $\tilde{\mathcal{S}}$ must be either $\vec{S}^+$ or $\vec{S}^-$. We will call these two generalized serial mechanisms serial+ and serial-, since they are closely related in spirit to the serial mechanism in Moulin-Shenker (1991).

Before proceeding to the statement and the proof of the characterization theorem for the two-good case, we first need to establish some notation and definitions. Since $\vec{S}(\vec{\lambda})$ is independent of $\|\lambda_i\|$, we can describe the utility functions with a single parameter. Define $\lambda(\theta) = (\cos\theta, \sin\theta)$, and then define θ_i as the unique value of θ such that $\lambda(\theta) = \frac{\lambda_i}{\|\lambda_i\|}$. We will use the notation $\vec{S}(\vec{\theta})$. Define $\theta_A(\vec{\theta})$ as the value of θ such that $\lambda(\theta_A(\vec{\theta}))$ is parallel to the vector $(\frac{\partial A}{\partial x^1}, \frac{\partial A}{\partial x^2})$ when the derivatives are evaluated at $\sum_{i=1}^n S_i(\vec{\theta})$.

Define the sets $\mathcal{X}(\vec{S})$ and $\mathcal{Z}(\vec{S})$ in the same way as before, except in terms of the $\vec{\theta}$.

$$\mathcal{X}(\vec{S}) \equiv \{\vec{\theta} \in (0, \frac{\pi}{2})|\vec{S}(\vec{\theta}) \in \Re^{2n}_{++}\}$$

$$\mathcal{Z}(\vec{S}) = \{\vec{\theta} \in \mathcal{X}(\vec{S})|i \not\approx j \ \forall \ i \neq j\}$$

We will say a social choice function $\vec{S}$ is *Z_2-smooth* if it has the following two properties:

1. The set $\mathcal{X}(\vec{S})$ is nonempty.
2. All $\vec{\theta} \in \mathcal{Z}(\vec{S})$ are differentiable points of $\vec{S}$ with $\frac{\partial S_i(\vec{\theta})}{\partial \theta_i} \neq 0$ for all i.

We say a social choice function $\vec{S}$ is *recusively Z_2-smooth* if $\vec{S}$ and all of the reduced forms $\vec{S}^{E_1;E_2;\ldots;E_{K-1};E_K}$, seen as functions from $[0, \frac{\pi}{2}]^K$ to $\Re^{2K}_{+}$, are all *Z_2-smooth.*

With these notational issues settled, we can now state the characterization theorem for the two-good case.

Theorem 3 *Every recursively Z_2-smooth and anonymous $\vec{S} \in \tilde{\mathcal{S}}$ is either the serial+ mechanism or the serial- mechanism.*

Proof: For convenience, the proof is partitioned into 7 steps. Assume that $\mathcal{Z}(\vec{S})$ is nonempty and then consider some $\vec{\theta} \in \mathcal{Z}(\vec{S})$. In step 6 we will show that $\mathcal{Z}(\vec{S})$ is nonempty.

Step 1 *The matrix $\frac{\partial S_i}{\partial \theta_j}(\vec{\theta})$ is acyclic.*

This result is merely a restatement of Theorem 4, which we discuss in Section 6 when we prove the characterization theorem for the general m case, and so we shall not repeat the proof here. For convenience, though, we will repeat the relevant definition. Consider some $n \times n$ matrix $a_{i,j}$. A *cycle* of length K, $K > 1$, of this matrix is a nonrepeating sequence of indices $[i_1, i_2, i_3, \ldots, i_{K-1}, i_K]$ such that $a_{i_K,i_1} \neq 0$ and $a_{i_j,i_{j+1}} \neq 0$ for all $1 \leq j < K$. We say that a matrix $a_{i,j}$ is *acyclic* if it has no cycles of length greater than one. For each square acyclic matrix, there must be an element j such that $a_{i,j} = 0$ for all $i \neq j$.

Step 2 $\frac{\frac{\partial S_i(\vec{\theta})}{\partial \theta_i}}{||\frac{\partial S_i(\vec{\theta})}{\partial \theta_i}||} = \lambda(\theta_i + \frac{\pi}{2})$

Note that $\frac{\partial \lambda}{\partial \theta}(\theta_i) = \lambda(\theta_i + \frac{\pi}{2})$, and that $\lambda(\theta_i) \cdot \lambda(\theta_i + \frac{\pi}{2}) = 0$. Since strategy-proofness requires $\lambda(\theta_i) \cdot [S_i(\vec{\theta}) - S_i(\vec{\theta}|^i\gamma)] \geq 0$ for all γ, differentiability then requires that (1) $\lambda(\theta_i) \cdot \frac{\partial S_i}{\partial \theta_i}(\vec{\theta}) = 0$ and (2) $\lambda(\theta_i) \cdot \frac{\partial^2 S_i}{(\partial \theta_i)^2}(\vec{\theta}) \leq 0$. Relation (1) specifies the direction of $\frac{\partial S_i}{\partial \theta_i}(\vec{\theta})$ up to a sign: $\frac{\partial S_i}{\partial \theta_i}(\vec{\theta}) = c\lambda(\theta_i + \frac{\pi}{2})$ for some scalar $c \neq 0$. Taking the derivative with respect to θ_i of relation (1) yields $\frac{\partial \lambda}{\partial \theta}(\theta_i) \cdot \frac{\partial S_i}{\partial \theta_i}(\vec{\theta}) + \lambda(\theta_i) \cdot \frac{\partial^2 S_i}{(\partial \theta_i)^2}(\vec{\theta}) = 0$, which in turn, through substitution of relation (2), yields $\frac{\partial \lambda}{\partial \theta}(\theta_i) \cdot \frac{\partial S_i}{\partial \theta_i}(\vec{\theta}) \geq 0$. This specifies the sign of $\frac{\partial S_i}{\partial \theta_i}(\vec{\theta})$, and so $\frac{\partial S_i}{\partial \theta_i}(\vec{\theta}) = c\lambda(\theta_i + \frac{\pi}{2})$ for some scalar $c > 0$. Note that this then implies that $\lambda(\theta_i) \cdot \frac{\partial^2 S_i}{(\partial \theta_i)^2}(\vec{\theta}) < 0$.

Step 3 *$S_i(\vec{\theta}) \neq S_i(\vec{\theta})$ for all $i \neq j$.*

Assume that there is some pair i, j such that $S_i(\vec{\theta}) = S_j(\vec{\theta})$. From the acyclicity result, we have either $\frac{\partial S_i}{\partial \theta_j}(\vec{\theta}) = 0$ or $\frac{\partial S_j}{\partial \theta_i}(\vec{\theta}) = 0$; we will assume, without loss of generality, that $\frac{\partial S_i}{\partial \theta_j}(\vec{\theta}) = 0$. Strategy-proofness implies $\lambda(\theta_j) \cdot \frac{\partial S_j}{\partial \theta_j}(\vec{\theta}) = 0$, and Step 2 requires that $\lambda(\theta_i) \cdot \frac{\partial S_j}{\partial \theta_j}(\vec{\theta}) = c\lambda(\theta_i) \cdot \lambda(\theta_j + \frac{\pi}{2}) = c\sin(\theta_i - \theta_j)$ for some $c > 0$. Thus, for sufficiently small $\epsilon > 0$, $\lambda(\theta_i) \cdot [S_i(\vec{\theta}|^j\theta_j + \epsilon(\theta_i - \theta_j)) - S_j(\vec{\theta}|^j\theta_j + \epsilon(\theta_i - \theta_j))] < 0$, indicating that agent i envies agent j. Since allocations are envy-free, we have a contradiction. Thus, the set $\mathcal{Z}(\vec{S}) = \{\vec{\theta} \in \mathcal{X}(\vec{S}) | S_i(\vec{\theta}) \neq S_j(\vec{\theta}) \; \forall \; i, j\}$.

Step 4 *Uniqueness of tail element*

The acyclicity condition stated in Step 1 requires that there must be at least one element, call it t, such that $\frac{\partial S_i}{\partial \theta_t}(\vec{\theta}) = 0$ for all $i \neq t$. For this element, $\frac{\partial S_t}{\partial \theta_t}(\vec{\theta})$ must be orthogonal to $(\frac{\partial A}{\partial x^1}, \frac{\partial A}{\partial x^2})$ evaluated at the total bundle $\sum_{i=1}^n S_i(\vec{\theta})$. Therefore, we must have $\theta_t = \theta_A(\vec{\theta})$. Since this applies to any tail element, and we are in $\mathcal{Z}(\vec{S})$, there can only be one such element.

Step 5 *Coalescing order*

Consider the tail element t, and any nontail element i such that there are no θ_j in between θ_i and θ_t. Consider the family of profiles $(\vec{\theta}|^t\epsilon\theta_i + (1-\epsilon)\theta_t)$ for $\epsilon \in [0,1]$. Let $\bar{\epsilon}$ denote the first value at which this profile leaves $\mathcal{Z}(\vec{S})$ as ϵ is increased from $\epsilon = 0$ towards $\epsilon = 1$. The set of profiles at which a particular element is a tail element is a closed set in $\mathcal{Z}(\vec{S})$, and there must be a unique tail element at each profile in $\mathcal{Z}(\vec{S})$; therefore, for the tail element to change as we continuously vary ϵ the profile $(\vec{\theta}|^t\epsilon\theta_i + (1-\epsilon)\theta_t)$ must leave $\mathcal{Z}(\vec{S})$. Accordingly, element t remains the unique tail element for all $\epsilon \in [0, \bar{\epsilon})$. Thus, $S_j(\vec{\theta}) = S_j(\vec{\theta}|^t\epsilon\theta_i + (1-\epsilon)\theta_t)$ for all $\epsilon \in [0, \bar{\epsilon}]$ and all $j \neq t$.

Note that the profile $(\vec{\theta}|^t\epsilon\theta_i + (1-\epsilon)\theta_t)$ remains in $\mathcal{X}(\vec{S})$ for all $\epsilon \in [0, \bar{\epsilon}]$. This follows trivially from the definitions for all $\epsilon \in [0, \bar{\epsilon})$, and we now establish that it holds for $\epsilon = \bar{\epsilon}$. If $S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) \notin \Re^m_{++}$ then $S^1_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) = 0$, $S^2_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) \neq 0$ if $\theta_i > \theta_t$ and $S^1_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) \neq 0$, $S^2_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) = 0$ if $\theta_i < \theta_t$. It is impossible to find any allocation other than $S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)$ that both lies on or above the indifference curve for $\lambda(\theta_i)$ at $S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)$ and also lies on or below the indifference curve for $\lambda(\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)$ at $S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)$, so either $\lambda(\theta_i) \cdot [S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) - S_i(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)] > 0$ or $\lambda(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) \cdot [S_i(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) - S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t)] > 0$. This violates the envy-free result, so the conclusion follows.

Thus, to leave $\mathcal{Z}(\vec{S})$, there must be some nontail element j such that $S_t(\vec{\theta}|^t\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) = S_j(\vec{\theta})$. Because equality implies coalescence (Step 3), we must have $S_j(\vec{\theta}) = S_j(\vec{\theta}|^{\{j,t\}}\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) = S_j(\vec{\theta}|^{\{j,t\}}\theta_j)$. However, if we consider the reduced form $\vec{S}^{\{j,t\}}$, we note that $(\vec{\theta}|^{\{j,t\}}\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t) \in \mathcal{Z}(\vec{S}^{\{j,t\}})$ and $(\vec{\theta}|^{\{j,t\}}\theta_j) \in \mathcal{Z}(\vec{S}^{\{j,t\}})$. The smoothness condition on $\vec{S}^{\{j,t\}}$ requires that $\frac{\partial S^{\{j,t\}}_{\{j,t\}}}{\partial \theta_{\{j,t\}}}(\vec{\theta}) \neq 0$, which contradicts our previously derived equality unless $\bar{\epsilon}\theta_i + (1-\bar{\epsilon})\theta_t = \theta_j$ (to see this, note that if they are not equal, then $\lambda(\theta_j)\cdot[S^{\{j,t\}}_{\{j,t\}}(\vec{\theta}|^{j,t}(1-\delta)(\bar{\epsilon}\theta_i+(1-\bar{\epsilon})\theta_t)+\delta\theta_j)-S^{\{j,t\}}_{\{j,t\}}(\vec{\theta}|^{j,t}\theta_j)]>0$ for small enough δ). We can then conclude that $\bar{\epsilon}=1$ and $j=i$. Therefore, for all $\epsilon\in[0,1)$, element t remains the unique tail element and thus, for all $j\neq t$ and $\epsilon\in[0,1]$, $S_j(\vec{\theta})=S_j(\vec{\theta}|^t\epsilon\theta_i+(1-\epsilon)\theta_t)$. Furthermore, $\theta_A(\vec{\theta}|^t\epsilon\theta_i+(1-\epsilon)\theta_t)=\epsilon\theta_i+(1-\epsilon)\theta_t$ for all $\epsilon\in[0,1]$. Consider the profiles $(\vec{\theta}|^j\theta_j+\delta)$ for any $j\neq i$. For sufficiently small δ, we know that element t is the unique tail, and we can repeat the above construction. By letting $\delta\to 0^+$, and taking differences, we can see that for all $\epsilon\in[0,1)$ and for all $j,k\neq t$, we must have $\frac{\partial S_k}{\partial\theta_j}(\vec{\theta})=\frac{\partial S_k}{\partial\theta_j}(\vec{\theta}|^t\epsilon\theta_i+(1-\epsilon)\theta_t)$.

At the profile $(\vec{\theta}|^{\{i,t\}}\theta_i)$, the element $\{i,t\}$ is the unique tail element of the reduced form $\vec{S}^{\{j,t\}}$. This follows because no other element $j\neq\{i,t\}$ has $\theta_j=\theta_A(\vec{\theta}|^{\{i,t\}}\theta_i)$. For all $\tilde{\theta}_i$ sufficiently close to θ_i and all $j\neq i,t$, we then have $S_j(\vec{\theta}|^{\{i,t\}}\tilde{\theta}_i)=S_j(\vec{\theta}|^{\{i,t\}}\theta_i)$. Furthermore, for all $\tilde{\theta}_i$ sufficiently close to θ_i, we could repeat the construction in the preceding paragraph to show that $S_j(\vec{\theta}|^{\{i,t\}}\tilde{\theta}_i)=S_j(\vec{\theta}|^i\tilde{\theta}_i)$. Therefore, $S_j(\vec{\theta}|^i\tilde{\theta}_i)=S_j(\vec{\theta}|^{\{i,t\}}\tilde{\theta}_i)=S_j(\vec{\theta}|^{\{i,t\}}\theta_i)=S_j(\vec{\theta})$ and so $\frac{\partial S_j}{\partial\theta_i}(\vec{\theta})=0$ for all $j\neq i,t$. Thus, we must have $\frac{\partial S_t}{\partial\theta_i}(\vec{\theta})\neq 0$ or else i would also be a tail element.

We can continue this construction iteratively, considering at each stage the reduced form induced by coalescing the tail element with some other element which has no θ in between. This yields a coalescing ordering. Label the elements accordingly, so that element n is the tail element of the original profile, and the coalescings proceed in order to $n-1$, $n-2$, etc.. Since $\frac{\partial S_{i+1}}{\partial\theta_i}(\vec{\theta})\neq 0$ for each $i<n$, this ordering must be unique (if there were two orderings, then the matrix $\frac{\partial S_i}{\partial\theta_j}(\vec{\theta})$ would not be acyclic). However, the construction could have produced any ordering where the θ's never crossed during the coalescing procedure. Thus, the θ's, when labelled with the tail element of every reduced form being the highest numbered uncoalesced element, must either be in strictly increasing order or in strictly decreasing order: i.e., either we have $\theta_i<\theta_{i+1}$ for all i, or we have $\theta_i>\theta_{i+1}$ for all i. Furthermore, in either case, $S_1(\vec{\theta})=S_1(\vec{\theta}|^N\theta_1)$, $S_2(\vec{\theta})=S_2(\vec{\theta}|^{N-1}\theta_2)$, $S_3(\vec{\theta})=S_3(\vec{\theta}|^{N-\{1,2\}}\theta_3)$, $\ldots$, $S_{n-1}(\vec{\theta})=S_{n-1}(\vec{\theta}|^{\{n,n-1\}}\theta_{n-1})$.

Step 6 *$\mathcal{Z}(\vec{S})$ nonempty*

Let $D(\vec{\theta})$ denote the number of distinct values of allocations $S_i(\vec{\theta})$; $\vec{\theta}\in\mathcal{Z}(\vec{S})\Leftrightarrow D(\vec{\theta})=n\ and\ \vec{\theta}\in\mathcal{X}(\vec{S})$. Assume, to the contrary, that $\mathcal{Z}(\vec{S})$ is empty. Consider some $\vec{\theta}\in\mathcal{X}(\vec{S})$ such that there is no other point in $\mathcal{X}(\vec{S})$ with a higher value for D, and let $K=D(\vec{\theta})<n$. We can now perform a reduction process that will produce a profile $\vec{R}(\vec{\theta})$ such that $\vec{S}(\vec{R}(\vec{\theta}))=\vec{S}(\vec{\theta})$ and $\vec{R}(\vec{\theta})\in\mathcal{Z}(\vec{S}^{E_1;E_2;\ldots;E_K})$ for some set of cliques E_i. The reduction process is as follows.

Consider each element to be its own clique. If there exist a pair of cliques i,j which can be coalesced, then we can consider the modified profile $(\vec{\theta}|^i\theta_j)$ instead without changing the allocations. Consider the pair i,j to be a clique $\{i,j\}$. The reduction process consists of iteratively coalescing cliques until there are no coalesceable pairs left. Define $\vec{R}(\vec{\theta})$ as the resulting profile and note that $\vec{S}(\vec{\theta})=\vec{S}(\vec{R}(\vec{\theta}))$. Let $E_1,E_2,\ldots,E_r$ denote the resulting cliques of elements. The resulting profile $\vec{R}(\vec{\theta})$ must be in $\mathcal{Z}(\vec{S}^{E_1;E_2;\ldots;E_r})$, since each remaining pair of cliques are not coalesced. As shown in Step 3, the allocations of the remaining cliques must all be distinct, and so $r=K$. For convenience, we will assume that $\vec{\theta}=\vec{R}(\vec{\theta})$ (we could have just as easily picked $\vec{R}(\vec{\theta})$ originally). Note that $\theta_i=\theta_j\Leftrightarrow i,j\in E_k\ for\ some\ k$ (because of the reducing process).

Consider any i for which $|E_i| > 1$ and let e_1 and e_2 be a partition of this clique: $e_1 \cup e_2 = E_i$ and $e_1 \cap e_2 = \emptyset$. For sufficiently small ϵ, $D(\vec{\theta}|^{e_1;e_2}\theta_{e_1} + \epsilon, \theta_{e_1} - \epsilon) = K+1$. This follows because $\vec{S}^{E_1;E_2;\ldots;E_K}_{E_i}(\vec{\theta}|^{E_i}\theta_{e_1} - \epsilon) \neq \vec{S}^{E_1;E_2;\ldots;E_K}_{E_i}(\vec{\theta}|^{E_i}\theta_{e_1} + \epsilon)$. We have reached a contradition, and so $\mathcal{Z}(\vec{S})$ must be nonempty. In fact, $\mathcal{Z}(\vec{S}) = \{\vec{\theta} \in \mathcal{X}(\vec{S}) | \theta_i \neq \theta_j \ for\ i \neq j\}$.

Step 7 $\vec{S} = \vec{S}^+$ *or* $\vec{S} = \vec{S}^-$

Let $b(s)$ be a C^2 parameterization of the production technology manifold, as described in the beginning of Section 5: $b : [0,1] \mapsto \Re^m_+$, $b^1(0) = 0$, $b^2(1) = 0$, and $b([0,1]) = \mathcal{P}$, with $b'(s) \neq 0$, $b^1(s)$ strictly increasing, and $b^2(s)$ strictly decreasing. At the end of Step 5, we were able to conclude that there are just two cases: (1) the coalescing numbering is the same as the order of increasing θ's, or (2) the coalescing numbering is the same as the order of decreasing θ's. Whichever case holds for one value of $\vec{\theta} \in \mathcal{Z}(\vec{S})$ must hold for all values of $\vec{\theta} \in \mathcal{Z}(\vec{S})$, since we can always find a linear path in $\mathcal{Z}(\vec{S})$ connecting one member of $\mathcal{Z}(\vec{S})$ to a permutation of any other member of $\mathcal{Z}(\vec{S})$, and the coalescing order must be unchanged along this path. We now show that in fact, the first case is just $\vec{S}^+$ and the second case is just $\vec{S}^-$.

Assume that we have the first case, where for any profile in $\mathcal{Z}(\vec{S})$ with $\theta_1 < \theta_2 \ldots < \theta_{n-1} < \theta_n$, the coalescing ordering is the same as the labelling and thus, by the results at the end of Step 5, we have: $S_1(\vec{\theta}) = S_1(\vec{\theta}|^N\theta_1)$, $S_2(\vec{\theta}) = S_2(\vec{\theta}|^{N-1}\theta_2)$, $S_3(\vec{\theta}) = S_3(\vec{\theta}|^{N-\{1,2\}}\theta_3)$, $\ldots$, $S_{n-1}(\vec{\theta}) = S_{n-1}(\vec{\theta}|^{\{n,n-1\}}\theta_{n-1})$.

We know that $\{S^N_N(\vec{\theta}|^N\gamma) | \gamma \in [0, \frac{\pi}{2}]\} = \{\frac{b(s)}{n} | s \in [0,1]\}$, since every point on $\mathcal{P}$ can be reached by setting γ so that the normal $(\frac{\partial A}{\partial x^1}, \frac{\partial A}{\partial x^2})$ is parallel to $\lambda(\gamma)$. Thus, if we parameterize the curve by arclength (and then linearly rescale so that the domain is still $[0,1]$), then the function remains C^2 and monotonic; we will, for convenience, also call this new function $b(s)$. Then, $b''(s) \in \Re^m_{--}$, since $b''(s)$ must be orthogonal to the tangent $b'(s)$ and thus antiparallel to the normal $(\frac{\partial A}{\partial x^1}, \frac{\partial A}{\partial x^2})$. Therefore, we can apply the serializing construction to it to produce a generalized serial function and its associated generalized serial mechanism. Now, define $\vec{t}$ by letting t_i be the unique maximizer of $\lambda(\theta_i) \cdot b(s)$. We must then have $S_1(\vec{\theta}) = \frac{b(t_1)}{n}$.

Furthermore, $\{S^{N-1}_{N-1}(\vec{\theta}|^{N-1}\gamma) | \gamma \in (\theta_1, \frac{\pi}{2})\} = \{S_1(\vec{\theta}) + \frac{b(s)-b(t_1)}{n-1} | s \in (t_1, \bar{t}_2)\}$ where $\bar{t}_2$ is defined to be the smallest value of $s \in [t_1, 1]$ such that $S_1(\vec{\theta}) + \frac{b(s)-b(t_1)}{n-1} \notin \Re^m_{++}$. Therefore, we must have $S_2(\vec{\theta}) = S_1(\vec{\theta}) + \frac{b(MIN[t_2,\bar{t}_2]) - b(t_1)}{n-1}$.

Repeating this reasoning, we then have that $S_1(\vec{\theta}) = \frac{b(t_1)}{n}$ and, for $1 < i \leq n$, $S_i(\vec{\theta}) = S_{i-1}(\vec{\theta}) + \frac{b(MIN[t_i,\bar{t}_i]) - b(MIN[t_{i-1},\bar{t}_{i-1}])}{n-i+1}$, where $\bar{t}_1 = 0$ and $\bar{t}_i$ for $i > 1$ is defined to be the smallest value of $s \in [\bar{t}_{i-1}, 1]$ such that $S_{i-1}(\vec{\theta}) + \frac{b(s) - b(MIN[t_{i-1},\bar{t}_{i-1}])}{n-i+1} \notin \Re^m_{++}$. This is exactly the application of the serializing construction to $b(s)$; therefore, $\vec{S} = \vec{S}^+$ if we are in the case that the coalescing numbering is the same as the order of increasing θ's. The same line of reasoning leads directly to the result that $\vec{S} = \vec{S}^-$ if we are in the case that the coalescing numbering is the same as the order of decreasing θ's (since reversing the labelling of the elements is equivalent to applying the same construction to $b(1-s)$). $\Box$

The $m = 2$ analogs of Corollaries 1 and 3 carry over directly to this case. The $m = 2$ analog of

Corollary 2 is much stronger.

Corollary 4 *For $m = 2$, there is a recursively Z_2-smooth and anonymous $\vec{S} \in \hat{\mathcal{S}}$ if and only if $\mathcal{P}$ is strictly convex.*

Proof: This *only if* result follows directly from the fact that $\lambda(\theta) \cdot b(s)$ has a unique maximizer for all $\theta \in [0, \frac{pi}{2}]$, and so $\mathcal{P}$ must be strictly convex. To show the *if* result, assume that $\mathcal{P}$ is strictly convex. Let $b(s)$ be a C^2 parameterization of the production technology manifold, as described in the beginning of Section 5: $b : [0,1] \mapsto \Re^m_+$, $b^1(0) = 0$, $b^2(1) = 0$, and $b([0,1]) = \mathcal{P}$, with $b'(s) \neq 0$, $b^1(s)$ strictly increasing, and $b^2(s)$ strictly decreasing. We must have $b''(s) \in \Re^2_-$ for all s. Apply the serializing construction to $b(s)$. It is clear that the function $\vec{a}(\vec{z})$ produced by the serializing construction satisfies conditions (1) and (4) in the definition of generalized serial functions. Furthermore, the opportunity sets $a_i(\vec{z}|^i s)$ are strictly convex, since $\mathcal{P}$ is, so conditions (2) and (3) are also satisfied. □

There is also one related result that does not have a general m counterpart. The simplicity of the two-good case allows us to generalize the results to a slightly different domain. Consider a case where the constraint functions are of the form $A(y) = 0$ where $y^1 = MAX[x^1_i]$ and $y^2 = \sum_{i=1}^n x^2_i$. This is similar to the excludable public goods problem considered in Moulin (1991). The above proofs generalize directly to this case, except that we need to replace the section about a unique last element. This can be shown by noting that the tail element must have maximal consumption of the first good; otherwise, it would increase its consumption of that good without decreasing its consumption of the other good. Once we establish uniqueness of tail element, the tail element sees a smooth constraint surface while the profile is in $\mathcal{Z}(\vec{S})$, and thus the proof goes through. Note that the kinks in the feasible set does not cause a problem, since we make no assumption about differentiability at points outside of $\mathcal{Z}(\vec{S})$.

Step 1′ *Unique Tail Element*

There must be at least one element, call it t, such that $\frac{\partial S_i}{\partial \theta_t}(\vec{\theta}) = 0$ for all $i \neq t$. For this element, $\frac{\partial S_t}{\partial \theta_t}(\vec{\theta})$ must be parallel to $(\frac{\partial A}{\partial y^1}\frac{\partial y^1}{\partial x^1_t}, \frac{\partial A}{\partial y^2})$. From our smoothness condition, $\frac{\partial S_t}{\partial \theta_t}(\vec{\theta}) \neq 0$, and from strategy-proofness we have $\lambda(\theta_t) \cdot \frac{\partial S_t}{\partial \theta_t}(\vec{\theta}) = 0$. Therefore, we cannot have $\frac{\partial y^1}{\partial x^1_t} = 0$, so the tail element must have maximal consumption of good 1. If two elements have the same allocation of good 1 they must have the same allocation of good 2 (by no envy and the strict monotonicity of the utilities in $\mathcal{Z}(\vec{S})$), and since no two elements have the same allocations in $\mathcal{Z}(\vec{S})$, there can only be one such element.$\diamond$

6 Proof of the Characterization Theorem and Related Corollaries

We first establish a version of the acyclicity result first derived by Satterthwaite and Sonnenschein (1981). We then use this to prove the characterization theorem and the related corollaries.

6.1 Acyclicity

Satterthwaite and Sonnenschein (1981) showed that nonbossy and strategy-proof social choice functions were acyclic at differentiable points. Here, we rederive their results in a somewhat simplified setting; because our assumptions about differentiability and utility domains are not completely consistent with theirs, we cannot simply cite their result. Our proof is very similar to theirs, except that the fact that we can restrict ourselves to linear profiles allows for a significant degree of simplification.

At a differentiable point $\vec{\lambda}$, define the two matrices $Z(\vec{\lambda})$ and $\tilde{Z}(\vec{\lambda})$ as follows. $Z_{i,j} = 1$ if $\frac{\partial S_i^\beta}{\partial \lambda_j^\alpha} \neq 0$ for at least one pair $\alpha, \beta \in M$; set $Z_{i,j} = 0$ otherwise. Similarly, $\tilde{Z}_{i,j} = 1$ if $\sum_{\beta=1}^m \lambda_i^\beta \frac{\partial S_i^\beta}{\partial \lambda_j^\alpha} \neq 0$ for at least one $\alpha \in M$; set $\tilde{Z}_{i,j} = 0$ otherwise. Clearly $Z_{i,j} = 0 \Rightarrow \tilde{Z}_{i,j} = 0$.

Consider some $n \times n$ matrix $a_{i,j}$. A *cycle* of length K, $K > 1$, of this matrix is a nonrepeating sequence of indices $[i_1, i_2, i_3, \ldots, i_{K-1}, i_K]$ such that $a_{i_K,i_1} \neq 0$ and $a_{i_j,i_{j+1}} \neq 0$ for all $1 \leq j < K$. We say that a matrix $a_{i,j}$ is *acyclic* if it has no cycles of length greater than one. For each square acyclic matrix, there must be an element j such that $a_{i,j} = 0$ for all $i \neq j$. For convenience, let $\chi : \iota(K) \mapsto \iota(K)$ be defined by $\chi(i) = i - 1$ for $1 < i \leq K$ and $\chi(1) = K$. Before stating and proving the main theorem on acyclicity, we first establish a lemma.

Lemma 3 *For each $\vec{S} \in \hat{\mathcal{S}}$ the matrix $\tilde{Z}(\vec{\lambda})$ is acyclic at every differentiable point.*

Proof: Assume that the lemma is false. Then, there is some $\vec{S} \in \hat{\mathcal{S}}$ with a differentiable point $\vec{\lambda}$ such that the matrix $\tilde{Z}(\vec{\lambda})$ has a cycle in it. Without loss of generality, assume that the shortest cycle is of length K, and consists of the elements $[1, 2, 3, \ldots, K-1, K]$; then, the only nonzero elements in the $K \times K$ submatrix $\tilde{Z}_{1 \leq i,j \leq K}$ are $\tilde{Z}_{(\chi(i),i)}$. Define $\vec{l} \in \Re^{nm}$ by $l_i = 0$ for $i > K$ and $l_i^\alpha = \sum_{\beta=1}^m \lambda_{\chi(i)}^\beta \frac{\partial S_{\chi(i)}^\beta}{\partial \lambda_i^\alpha}$ for $i \leq K$. Thus, $\lambda_j \cdot \sum_{\alpha=1}^m \frac{\partial S_j}{\partial \lambda_i^\alpha} l_i^\alpha > 0$ if $j = \chi(i)$ and $i \leq K$; the sum $\lambda_j \cdot \sum_{\alpha=1}^m \frac{\partial S_j}{\partial \lambda_i^\alpha} l_i^\alpha = 0$ for all other combinations of i, j with $i, j \leq K$. One can then find a positive constant ϵ such that $\lambda_i \cdot S_i(\vec{\lambda} + \epsilon \vec{l}) > \lambda_i \cdot S_i(\vec{\lambda})$ for each $1 \leq i \leq K$. This contradicts the fact that every $\vec{S} \in \hat{\mathcal{S}}$ is WCSP. □

Theorem 4 *For each $\vec{S} \in \hat{\mathcal{S}}$ the matrix $Z(\vec{\lambda})$ is acyclic at every differentiable point.*

Proof: Given Lemma 3, if the above theorem is not true there must be a differentiable point where Z has a cycle where $\tilde{Z}$ does not. We now show that this is impossible. Note that since the derivatives are continuous in a neighborhood $\mathcal{N}$ of the differentiable point, the set of profiles in $\mathcal{N}$ at which Z has a cycle is an open subset of $\mathcal{N}$; similarly, the set of profiles in $\mathcal{N}$ at which $\tilde{Z}$ has a cycle is a subset of the points at which Z has a cycle, and is also an open subset of $\mathcal{N}$. Thus, the set of points in $\mathcal{N}$ at which $\tilde{Z}$ is acyclic but Z has a cycle must either be empty or have a nonempty interior in $\mathcal{N}$. Assume that this set is nonempty, and pick some interior point where Z has a cycle where $\tilde{Z}$ does not. Then, there is some open neighborhood $\mathcal{M}$ and some pair i, j such that $Z_{i,j} = 1$ but $\tilde{Z}_{i,j} = 0$ in $\mathcal{M}$. Consider some $l_j \in \Re^m$ such that $\sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_j^\alpha} l_j^\alpha \neq 0$. Note that at any $\vec{\lambda} \in \mathcal{M}$, and for any $l_i \in \Re^m$, we must have

$$\lambda_i \cdot \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_i^\alpha} l_i^\alpha = 0$$

and

$$\lambda_i \cdot \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_j^\alpha} l_j^\alpha = 0$$

The first result follows from the strategy-proof condition. The second follows from the assumption that $\tilde{Z}_{i,j} = 0$. Varying λ_j by l_j in the first equation, we find:

$$\lambda_i \cdot \sum_{\beta=1}^m \sum_{\alpha=1}^m \frac{\partial^2 S_i}{\partial \lambda_j^\beta \partial \lambda_i^\alpha} l_i^\alpha l_j^\beta = 0$$

Varying λ_i by l_i in the second equation, we find:

$$\lambda_i \cdot \sum_{\beta=1}^m \sum_{\alpha=1}^m \frac{\partial^2 S_i}{\partial \lambda_j^\beta \partial \lambda_i^\alpha} l_i^\alpha l_j^\beta + l_i \cdot \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_j^\alpha} l_j^\alpha = 0$$

Choose $l_i = \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_j^\alpha} l_j^\alpha$, so the second term is nonzero. These two equations are in contradiction, thereby completing the proof of the theorem. □

6.2 Proof of Characterization Theorem

Proof: For convenience, this proof is partitioned into 10 steps.

Step 1 *Definition of* $\mathcal{Y}(\vec{S})$

We can define a relation that is similar to, but strictly weaker than, the *coalescence* relation $\approx$; we say that elements i, j can *partially coalesce* if $S_j(\vec{\lambda}) = S_i(\vec{\lambda})$ and either $S_i(\vec{\lambda}) = S_i(\vec{\lambda}|^i\lambda_j)$ or $S_j(\vec{\lambda}) = S_j(\vec{\lambda}|^j\lambda_i)$. We denote this reflexive and symmetric relation by $i \sim j$, and its negation by $i \not\sim j$. Define the set $\mathcal{Y}(\vec{S})$:

$$\mathcal{Y}(\vec{S}) = \{\vec{\lambda} \in \mathcal{X}(\vec{S}) | i \not\sim j \ \forall \ i \neq j\}$$

Clearly, $\mathcal{Y}(\vec{S}) \subseteq \mathcal{Z}(\vec{S})$. Assume that the set $\mathcal{Y}(\vec{S})$ is nonempty, and consider some point $\vec{\lambda} \in \mathcal{Y}(\vec{S})$. Later, in Step 6, we will show that indeed $\mathcal{Y}(\vec{S})$ is nonempty.

Step 2 *Definition of vector spaces.*

Define the tangent space $TS_i(\vec{\lambda})$ as the vector space spanned by the m vectors $\frac{\partial S_i}{\partial \lambda_i^\alpha}$ (one for each $\alpha \in M$). Also, define $NS_i(\vec{\lambda})$ as the vector space normal to $TS_i(\vec{\lambda})$. Define $NL_i(\vec{\lambda})$ as the vector space spanned by all $l_i \in \Re^m$ such that $\sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_i^\alpha} l_i^\alpha = 0$, and $TL_i(\vec{\lambda})$ as the vector space normal to

$NL_i(\vec{\lambda})$. Since strategy-proofness requires that $\lambda_i \cdot \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_i^\alpha} l_i^\alpha = 0$ for all $l_i \in \Re^m$, we must have $\lambda_i \in NS_i(\vec{\lambda})$. Furthermore, since $\vec{S}(\vec{\lambda}|^i c\lambda_i) = \vec{S}(\vec{\lambda})$ for any $c > 0$, we have $\lambda_i \in NL_i(\vec{\lambda})$. As with any linear transformation, $n = dim(TS_i(\vec{\lambda})) + dim(NS_i(\vec{\lambda})) = dim(TL_i(\vec{\lambda})) + dim(NL_i(\vec{\lambda})) = dim(TS_i(\vec{\lambda})) + dim(NL_i(\vec{\lambda}))$ (the last being the only nontrivial assertion), so $dim(NS_i(\vec{\lambda})) = dim(NL_i(\vec{\lambda}))$. Assume that there is some $l \in \Re^m$ such that $l \in NL_i(\vec{\lambda})$ but $l \notin NS_i(\vec{\lambda})$. Then, there is some $v \in \Re^m$ such that $l \cdot \sum_{\alpha=1}^m \frac{\partial S_i}{\partial \lambda_i^\alpha} v^\alpha > 0$. The smoothness condition requires that $S_i(\vec{\lambda}|^i \lambda_i + \epsilon l) = S_i(\vec{\lambda})$ for all ϵ such that $\lambda_i + \epsilon l \in \Re^m_{++}$. However, given some $\epsilon > 0$ there is a sufficiently small $\delta > 0$ such that $(\lambda_i + \epsilon l) \cdot [S_i(\vec{\lambda}|^i \lambda_i + \epsilon l) - S_i(\vec{\lambda}|^i \lambda_i + \delta v)] < 0$ violating strategy-proofness. Thus, we cannot find any $l \in \Re^m$ such that $l \in NL_i(\vec{\lambda})$ but $l \notin NS_i(\vec{\lambda})$, and so we can then conclude that $NL_i(\vec{\lambda}) \subseteq NS_i(\vec{\lambda})$. Therefore, $NS_i(\vec{\lambda}) = NL_i(\vec{\lambda})$ and so $TS_i(\vec{\lambda}) = TL_i(\vec{\lambda})$.

Since $\lambda_i \in NS_i(\vec{\lambda})$, we know that $\vec{S}(\vec{\lambda}|^i \tilde{\lambda}_i) = \vec{S}(\vec{\lambda})$ for any $\tilde{\lambda}_i \in NS_i(\vec{\lambda}) \cap \Re^m_{++}$. For any $v \in NS_i(\vec{\lambda})$, $\vec{S}(\vec{\lambda}|^i \tilde{\lambda}_i + v) = \vec{S}(\vec{\lambda}|^i \lambda_i + w)$ where $w = v + \tilde{\lambda}_i - \lambda_i \in NS_i(\vec{\lambda})$, and thus $\vec{S}(\vec{\lambda}|^i \tilde{\lambda}_i + v) = \vec{S}(\vec{\lambda}|^i \tilde{\lambda}_i)$. This implies that $NS_i(\vec{\lambda}|^i \tilde{\lambda}_i) = NS_i(\vec{\lambda})$ and thus $TS_i(\vec{\lambda}|^i \tilde{\lambda}_i) = TS_i(\vec{\lambda})$ for any $\tilde{\lambda}_i \in NS_i(\vec{\lambda}) \cap \Re^m_{++}$.

We can similarly define the space $TA(\vec{\lambda})$ as the null vector space of the matrix $\frac{\partial A^\nu}{\partial x^\alpha}$ evaluated at $\sum_{i=1}^n S_i(\vec{\lambda})$. This vector space has, by the assumption on the constraint functions, dimension $(m-p)$. Note that the entries of the matrix $\frac{\partial A^\nu}{\partial x^\alpha}$ are all positive, due to the strict monotonicity condition on the A^ν; thus, $x \in \{\Re^m_+ - 0\} \Rightarrow x \notin TA(\vec{\lambda})$. Also, define $NA(\vec{\lambda})$ as the vector space normal to $TA(\vec{\lambda})$. Since we have assumed $m > p > 0$, both of these vector spaces have nonzero dimension.

Step 3 *$S_i(\vec{\lambda}) \neq S_j(\vec{\lambda})$ for all $i \neq j$.*

Assume that there exists a pair i, j such that $S_i(\vec{\lambda}) = S_j(\vec{\lambda})$. From the assumption that $\vec{\lambda} \in \mathcal{Y}(\vec{S})$, we know that $i \not\sim j$ and so must have both $\vec{S}(\vec{\lambda}|^i \lambda_j) \neq \vec{S}(\vec{\lambda})$ and $\vec{S}(\vec{\lambda}|^j \lambda_i) \neq \vec{S}(\vec{\lambda})$. Defining $v \equiv (\lambda_i - \lambda_j)$, we know that $v \notin NS_i(\vec{\lambda})$ and $v \notin NS_j(\vec{\lambda})$. From the acyclicity result, we have either $Z_{i,j} = 0$ or $Z_{j,i} = 0$; we will assume, without loss of generality, that $Z_{i,j} = 0$. Thus, $\frac{\partial S_i}{\partial \lambda_j^\alpha} = 0$ for all $\alpha \in M$. Define $g(\epsilon) = S_j(\vec{\lambda}|^j \lambda_j + \epsilon v)$. From the smoothness condition, we have $v \cdot g'(0) > 0$. We also have $\lambda_j \cdot g'(0) = 0$, so $\lambda_i \cdot g'(0) > 0$. Thus, for sufficiently small ϵ, $\lambda_i \cdot [S_i(\vec{\lambda}|^j \lambda_j + \epsilon v) - S_j(\vec{\lambda}|^j \lambda_j + \epsilon v)] < 0$, indicating that agent i envies agent j. Since allocations are envy-free by Lemma 2, we have a contradiction. Thus, the set $\mathcal{Y}(\vec{S}) = \{\vec{\lambda} \in \mathcal{X}(\vec{S}) | S_i(\vec{\lambda}) \neq S_j(\vec{\lambda}) \ \forall\ i \neq j\}$.

Consider, for future use, any profile $\vec{\lambda} \in \mathcal{Z}(\vec{S})$. The arguments in the preceding paragraph show that if $S_i(\vec{\lambda}) = S_j(\vec{\lambda})$ and $Z_{i,j}(\vec{\lambda}) = 0$, then $\vec{S}(\vec{\lambda}|^j \lambda_i) = \vec{S}(\vec{\lambda})$. This follows from the fact that if $v = (\lambda_i - \lambda_j) \notin NS_j(\vec{\lambda})$, then at the profile $(\vec{\lambda}|^j \lambda_j + \epsilon v)$ element i will envy element j for small enough $\epsilon > 0$.

Step 4 *Uniqueness of tail element*

From Theorem 4, we know that there must be an element, call it t, such that $\frac{\partial S_i}{\partial \lambda_t^\alpha} = 0$ for all $i \neq t$ and all α. We call this element the *tail* element, and have $TS_t(\vec{\lambda}) \subseteq TA(\vec{\lambda})$ and thus $NA(\vec{\lambda}) \subseteq NS_t(\vec{\lambda})$. Let λ_{NA} be any element of $NA(\vec{\lambda}) \cap \Re^m_{++}$; then, $\vec{S}(\vec{\lambda}|^t\lambda_{NA}) = \vec{S}(\vec{\lambda})$ and $TA(\vec{\lambda}|^t\lambda_{NA}) = TA(\vec{\lambda})$. Thus, $(\vec{\lambda}|^t\lambda_{NA}) \in \mathcal{Y}(\vec{S})$. We now show that t is the unique tail element at the profile $(\vec{\lambda}|^t\lambda_{NA})$. If there were an element $i \neq t$ with $TS_i(\vec{\lambda}|^t\lambda_{NA}) \subseteq TA(\vec{\lambda}|^t\lambda_{NA})$ then $\vec{S}(\vec{\lambda}|^{\{i,t\}}\lambda_{NA}) = \vec{S}(\vec{\lambda}|^t\lambda_{NA}) = \vec{S}(\vec{\lambda})$, and so $S_i(\vec{\lambda}) = S_t(\vec{\lambda})$ contrary to the assumption that $\vec{\lambda} \in \mathcal{Y}(\vec{S})$. Since any other tail element $i \neq t$ must have $TS_i(\vec{\lambda}|^t\lambda_{NA}) \subseteq TA(\vec{\lambda}|^t\lambda_{NA})$, and at the profile $(\vec{\lambda}|^t\lambda_{NA})$ no element besides t has this property, element t is the only tail element at this profile $(\vec{\lambda}|^t\lambda_{NA})$. We will need two facts in the paragraphs that follow. First, at any profile $\vec{\gamma} \in \mathcal{Y}(S)$ with a unique tail element t, element t is the unique tail element for all sufficiently nearby profiles and thus $\frac{\partial^2 S_i}{\partial\lambda_t^\alpha\partial\lambda_i^\beta}(\vec{\gamma}) = 0$ for all $i \neq t$. Second, if some profile $\vec{\gamma} \in \mathcal{Y}(S)$ has only one element t such that $TS_t(\vec{\gamma}) \subseteq TA(\vec{\gamma})$, then element t is the unique tail element at this profile.

Consider the family of profiles $(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA})$ as we vary ϵ from 0 to 1. Since the allocations are constant along this path, $TA(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA}) = TA(\vec{\lambda}|^t\lambda_{NA})$. Assume that there is some $j \neq t$ and $\bar{\epsilon}$, $0 < \bar{\epsilon} \leq 1$, such that (1) $TS_j(\vec{\lambda}|^t\bar{\epsilon}\lambda_t + (1-\bar{\epsilon})\lambda_{NA}) \subseteq TA(\vec{\lambda}|^t\lambda_{NA})$, and (2) $TS_i(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA}) \not\subseteq TA(\vec{\lambda}|^t\lambda_{NA})$ for all $i \neq t$ and all $0 \leq \epsilon < \bar{\epsilon}$. For all $0 \leq \epsilon < \bar{\epsilon}$ t is the unique tail element and thus $\frac{\partial^2 S_i}{\partial\lambda_t^\alpha\partial\lambda_i^\beta}(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA}) = 0$ for all all $i \neq t$; therefore, $\frac{\partial S_i}{\partial\lambda_i^\beta}(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA}) = \frac{\partial S_i}{\partial\lambda_i^\beta}(\vec{\lambda}|^t\lambda_{NA})$. Consequently, $TS_i(\vec{\lambda}|^t\bar{\epsilon}\lambda_t + (1-\bar{\epsilon})\lambda_{NA}) = TS_i(\vec{\lambda}|^t\lambda_{NA}) \not\subseteq TA(\vec{\lambda}|^t\lambda_{NA})$ for all $i \neq t$. This contradicts our assumption that $TS_j(\vec{\lambda}|^t\bar{\epsilon}\lambda_t + (1-\bar{\epsilon})\lambda_{NA}) \subseteq TA(\vec{\lambda}|^t\lambda_{NA})$, thereby showing that for all $\epsilon \in [0,1]$, $TS_i(\vec{\lambda}|^t\epsilon\lambda_t + (1-\epsilon)\lambda_{NA}) \not\subseteq TA(\vec{\lambda}|^t\lambda_{NA})$ for all $i \neq t$. Therefore, t is the unique tail element at the original profile $\vec{\lambda}$.

This shows that at each profile in $\mathcal{Y}(\vec{S})$ there is exactly one tail element t. Furthermore, $TS_j(\vec{\lambda}) = TS_j(\vec{\lambda}|^t\lambda_{NA})$ for any $\lambda_{NA} \in NA(\vec{\lambda}) \cap \Re^m_{++}$ and any $j \in N$.

Step 5 *Coalescing order*

Since the set of points at which an element is the tail element is closed in $\mathcal{Y}(\vec{S})$, and there is always exactly one tail element, the only way the tail element can change as we continuously change $\vec{\lambda}$ is if the profile leaves $\mathcal{Y}(\vec{S})$. Consider a point in $\vec{\lambda} \in \mathcal{Y}(\vec{S})$ and, for some nontail element i, the path of profiles $(\vec{\lambda}|^t\lambda_t + \epsilon(\lambda_i - \lambda_t))$ as we vary ϵ from $\epsilon = 0$ to $\epsilon = 1$. Let $\bar{\epsilon}$ be the first value of ϵ such that the profile $(\vec{\lambda}|^t\lambda_t + \epsilon(\lambda_i - \lambda_t)) \notin \mathcal{Y}(\vec{S})$. Since $(\vec{\lambda}|^t\lambda_i) \notin \mathcal{Y}(\vec{S})$ (because the i'th and t'th allocations are the same), $\bar{\epsilon} \leq 1$. Furthermore, $S_k(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) = S_k(\vec{\lambda})$ for all $k \neq t$. There are two possibilities for leaving $\mathcal{Y}(\vec{S})$: (1) $(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) \notin \mathcal{X}(\vec{S})$, and (2) $(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) \in \mathcal{X}(\vec{S}) - \mathcal{Y}(\vec{S})$. Assume that possibility (1) occurs, so $S_t(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) \notin \Re^m_{++}$. From the smoothness requirement, we know that $\vec{S}(\vec{\lambda}|^t\lambda_t + \epsilon(\lambda_i - \lambda_t)) = \vec{S}(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t))$ for all $\epsilon > \bar{\epsilon}$ such that $\lambda_t + \epsilon(\lambda_i - \lambda_t) \in \Re^m_+$. In particular, $\vec{S}(\vec{\lambda}|^t\lambda_i) = \vec{S}(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t))$. However, we know that $S_t(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) = S_t(\vec{\lambda}|^t\lambda_i) = S_i(\vec{\lambda}|^t\lambda_i) = S_i(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) = S_i(\vec{\lambda}) \in \Re^m_{++}$; contradiction. Therefore, possibility (2) must hold; there must be some nontail element $j \neq t$ such that $S_t(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) = S_j(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t)) = S_j(\vec{\lambda})$.

Consider the extension to this path of profiles by looking at the segment going from $(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t))$ to $(\vec{\lambda}|^t\lambda_j)$. We claim that the allocations are constant along this path. If $t \approx j$ at $(\vec{\lambda}|^t\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t))$ then this follows immediately (because $\vec{S}$ is SSP, as we vary λ along any linear path the allocations are either constant or nonrepeating). If $t \sim j$ but $t \not\approx j$, the observation at the end of Step 3 implies that the allocations are constant along this path (if $(\lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t) - \lambda_j) \notin NS_t(\vec{\lambda}|^t \lambda_t + \bar{\epsilon}(\lambda_i - \lambda_t))$, then we could find a profile where j envied t).

We consider $\bar{\epsilon}$ as fixed, and now consider varying the profile. In the paragraphs that follow, let $\vec{\gamma}$ be any profile with $\|\vec{\gamma} - \vec{\lambda}\| < \delta$ for some sufficiently small $\delta > 0$. First, we can pick δ so that we know $\vec{\gamma} \in \mathcal{Y}(\vec{S})$ and that element t is the tail element. Next, consider the path of profiles $(\vec{\gamma}|^t \lambda)$ where λ first varies from γ_t to $\gamma_t + \bar{\epsilon}(\gamma_i - \gamma_t)$, and then varies from $\gamma_t + \bar{\epsilon}(\gamma_i - \gamma_t)$ to γ_j. We will define the path $\Upsilon(\vec{\gamma})$ as the portion of the path connected to the starting point which remains within $\mathcal{Y}(\vec{S})$. The endpoint of $\Upsilon(\vec{\gamma})$, denoted by $(\vec{\gamma}|^t \bar{\lambda})$, is the first point on the path outside of $\mathcal{Y}(\vec{S})$. Clearly such an endpoint must exist, because when $\lambda = \gamma_j$ the profile is no longer in $\mathcal{Y}(\vec{S})$. If δ is sufficiently small, then element t must be the unique tail element at the profile $\vec{\gamma}$, and $S_j(\vec{\gamma}|^t \bar{\lambda}) = S_t(\vec{\gamma}|^t \bar{\lambda})$ because element t could not have coalesced with any other element along the path (because the orignal path with $\vec{\lambda}$ was a finite distance away from any other allocation, and the full path is continuous in $\vec{\gamma}$).

We now claim that $\vec{S}(\vec{\gamma}|^t \bar{\lambda}) = \vec{S}(\vec{\gamma}|^t \gamma_j)$. At the endpoint of $\Upsilon(\vec{\gamma})$, element t has either fully coalesced with j, in which case the claim is clearly true, or element t has only partially coalesced with j, which is the case we deal with now. Assume that at $(\vec{\gamma}|^t \bar{\lambda})$, $t{\sim}j$ but $t{\not\approx}j$. Then, $(\vec{\gamma}|^t \bar{\lambda}) \in \mathcal{Z}(\vec{S})$ and element t is still a tail element so $Z_{j,t}(\vec{\gamma}|^t \bar{\lambda}) = 0$. We can then apply the observation made at the end of Step 3 to conclude that indeed, $\vec{S}(\vec{\gamma}|^t \bar{\lambda}) = \vec{S}(\vec{\gamma}|^t \gamma_j)$.

Therefore, $S_k(\vec{\gamma}|^t \gamma_j) = S_k(\vec{\gamma})$ for all $k \neq t$, and thus $S_k(\vec{\gamma}|^t \gamma_j) \neq S_t(\vec{\gamma}|^t \gamma_j)$ for all $k \neq t, j$. For each profile $(\vec{\lambda}|^t \lambda)$ in $\Upsilon(\vec{\lambda})$, $\vec{S}(\vec{\lambda}|^t \lambda) = \vec{S}(\vec{\lambda}|^t \tilde{\lambda})$ for all $\tilde{\lambda} \in NA(\vec{\lambda}|^t \lambda) \cap \Re^m_{++}$. Since $NA(\vec{\lambda}|^t \lambda)$ is continuous in λ, we can conclude that $S_t(\vec{\lambda}|^t \bar{\lambda}) = S_k(\vec{\lambda})$ for each element $k \neq t$ such that $NA(\vec{\lambda}|^t \bar{\lambda}) \subseteq NS_k(\vec{\lambda})$. Therefore, $NA(\vec{\lambda}|^t \bar{\lambda}) \nsubseteq NS_k(\vec{\lambda})$ for each $k \neq j, t$. Furthermore, $TS_k(\vec{\lambda}) \nsubseteq TA(\vec{\lambda}|^t \bar{\lambda})$ for each $k \neq j, t$.

Consider the reduced form $\vec{S}^{\{j,t\}}$. We know that $(\vec{\lambda}|^t \lambda_j) \in \mathcal{Y}(\vec{S}^{\{j,t\}})$. The fact that $S_k(\vec{\gamma}|^t \gamma_j) = S_k(\vec{\gamma})$ for all $k \neq t$ holds for all nearby $\vec{\gamma}$ implies that $\frac{\partial S_h^{\{j,t\}}}{\partial \lambda_k^\alpha}(\vec{\gamma}|^t \gamma_j) = \frac{\partial S_h}{\partial \lambda_k^\alpha}(\vec{\gamma})$ for all $h, k \neq t$. In particular, $TS_h^{\{j,t\}}(\vec{\gamma}|^t \gamma_j) = TS_h(\vec{\gamma})$ for all $h \neq t$. The fact that $TS_k(\vec{\lambda}) \nsubseteq NA(\vec{\lambda}|^t \bar{\lambda})$ for each $k \neq j, t$ implies that the element $\{j, t\}$ must be the unique tail of the reduced form $\vec{S}^{\{j,t\}}$ at the profile $(\vec{\lambda}|^t \lambda_j)$. Thus, $\frac{\partial S_k}{\partial \lambda_j^\alpha}(\vec{\lambda}) = 0$ for all $k \neq j, t$ and all $\alpha \in M$. Since we know that t is the unique tail element at the profile $\vec{\lambda}$, we must have $\frac{\partial S_t}{\partial \lambda_j^\alpha}(\vec{\lambda}) \neq 0$ for at least one α.

The conclusions of steps 1-4 and those derived so far in step 5 depended only on the weak anonymity of $\vec{S}$. Thus, we can apply these conclusions to the reduced form $\vec{S}^{\{j,t\}}$, finding an element k such that the element $\{j, t\}$ of the reduced form remains the unique tail element along a path until the elements k and $\{j, t\}$ coalesce. We can then, in turn, make the same arguments about the reduced form $\vec{S}^{\{k,j,t\}}$. Applying this process iteratively, we can therefore find an ordering, dubbed the *coalescing order*, of the elements. Without loss of generality, let the elements be ordered so that the highest numbered element is always the tail element. Furthermore, in this ordering we know that $Z_{i+1,i} \neq 0$ for $1 \leq i < n$. There is no other coalescing order since, if there were, the matrix $Z_{i,j}$ would have a cycle.

Returning to the original profile, with the labelling done as described above, consider the path $(\vec{\lambda}|^n\lambda_n + \epsilon(\lambda_{n-1} - \lambda_n))$ for $0 \leq \epsilon \leq 1$. Clearly $(\lambda_{n-1} - \lambda_n) \notin NS_n(\vec{\lambda})$ and $(\lambda_{n-1} - \lambda_n) \notin NS_{n-1}(\vec{\lambda})$. Again, let $\bar{\epsilon}$ denote the first value at which this profile leaves $\mathcal{Y}(\vec{S})$. Since there can be no other coalescing order, we must have $S_n(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = S_{n-1}(\vec{\lambda}) = S_{n-1}(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n))$. Elements n and $n-1$ must be fully coalesced at this profile. To see this, assume otherwise. Since $Z_{n-1,n}(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = 0$, we must have (following the observation at the end of Step 3) $\vec{S}(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = \vec{S}(\vec{\lambda}|^n\lambda_{n-1})$, and so $\vec{S}(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = \vec{S}(\vec{\lambda}|^n\lambda_n + \epsilon(\lambda_{n-1} - \lambda_n))$ for all ϵ such that $\lambda_n + \epsilon(\lambda_{n-1} - \lambda_n) \in \Re^m_{++}$. This is in contradiction to the fact that $(\lambda_{n-1} - \lambda_n) \notin NS_n(\vec{\lambda})$. Therefore, the two elements n and $n-1$ are fully coalesced at the profile $(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n))$. Thus, $\vec{S}(\vec{\lambda}|^n\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = \vec{S}(\vec{\lambda}|^{\{n,n-1\}}\lambda_{n-1}) = \vec{S}(\vec{\lambda}|^{\{n,n-1\}}\lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n))$. This can only occur if $\bar{\epsilon} = 1$, since $(\lambda_{n-1} - \lambda_n) \notin NS_{n-1}(\vec{\lambda})$.

Define $T(i) = \{i, i+1, \ldots, n-1, n\} = N - \iota(i-1)$. Applying this reasoning above recursively, we see that as we consider the profile path $(\vec{\lambda}|^{T(i)}\lambda_i + \epsilon(\lambda_{i-1} - \lambda_i))$, the tail clique $T(i)$ fully coalesces with the previous element $i-1$ when $\epsilon = 1$. Thus, $S_i(\vec{\lambda}) = S_i(\vec{\lambda}|^{T(i)}\lambda_i)$ and $TS_i^{1;2;\ldots;T(i)}(\vec{\lambda}|^{T(i)}\lambda_i) = TS_i(\vec{\lambda})$. We have also shown that $(\lambda_i - \lambda_{i+1}) \notin NS_i(\vec{\lambda})$ and $(\lambda_i - \lambda_{i+1}) \notin NS_{i+1}(\vec{\lambda})$ for each $i < n$, so the $TS_i(\vec{\lambda})$ must all have nonzero dimension.

Step 6 $\mathcal{Y}(\vec{S})$ *and* $\vec{S}(\mathcal{X}(\vec{S})) \in \overline{\vec{S}(\mathcal{Y}(\vec{S}))}$

Consider any point $\vec{\lambda} \in \mathcal{X}(\vec{S})$ which, from the smoothness condition, we know to be nonempty. If the allocations $S_i(\vec{\lambda})$ are all distinct, then the profile is in $\mathcal{Y}(\vec{S})$. If not, we can perform a reduction process that will produce a profile $\vec{R}(\vec{\lambda})$ such that $\vec{S}(\vec{R}(\vec{\lambda})) = \vec{S}(\vec{\lambda})$ and $\vec{R}(\vec{\lambda}) \in \mathcal{Y}(\vec{S}^{E_1;E_2;\ldots;E_r})$ for some set of cliques E_i. The reduction process is as follows.

Consider each element to be its own clique. If there exist a pair of cliques i, j where $S_i(\vec{\lambda}) = S_j(\vec{\lambda})$ and $S_i(\vec{\lambda}) = S_i(\vec{\lambda}|^i\lambda_j)$, then consider the modified profile $(\vec{\lambda}|^i\lambda_j)$ instead, and consider the pair i, j to be a clique. The reduction process consists of iteratively repeating this process of coalescing cliques until there are no such pairs left. Define $\vec{R}(\vec{\lambda})$ as the resulting profile. Let $E_1, E_2, \ldots, E_r$ denote the cliques of elements with identical $R_i(\vec{\lambda})$. The resulting profile $\vec{R}(\vec{\lambda})$ must be in $\mathcal{Y}(S^{E_1;E_2;\ldots;E_r})$, since each remaining pair of cliques are neither fully nor partially coalesced. As shown in Step 3, the allocations of the remaining cliques must all be distinct.

Let $D(\vec{\lambda})$ denote the number of distinct values of allocations $S_i(\vec{\lambda})$; $\vec{\lambda} \in \mathcal{Y}(\vec{S}) \Leftrightarrow D(\vec{\lambda}) = n$ *and* $\vec{\lambda} \in \mathcal{X}(\vec{S})$. We now make the following two claims.

Claim: For any $\vec{\lambda} \in \mathcal{X}(\vec{S})$, if $1 < D(\vec{\lambda}) < n$ then for any $\delta > 0$ there exists some $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta$ and $D(\vec{\eta}) > D(\vec{\lambda})$.

Proof: Assume, to the contrary, that there is some $\vec{\lambda} \in \mathcal{X}(\vec{S})$ and $\delta > 0$ such that $1 < D(\vec{\lambda}) < n$ and $D(\vec{\eta}) \leq D(\vec{\lambda})$ for all $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta$. For ease of notation, set $K = D(\vec{\lambda})$. We can, without loss of generality, assume that $\vec{\lambda} = \vec{R}(\vec{\lambda})$, and let us denote the cliques of $\vec{\lambda}$ by $E_1, E_2, \ldots, E_K$. Then, we know that $\vec{\lambda} \in \mathcal{Y}(\vec{S}^{E_1;E_2;\ldots;E_K})$. Since $\mathcal{Y}(\vec{S}^{E_1;E_2;\ldots;E_K})$ is nonempty, we can apply the conclusions of steps 1-5 to the reduced form $\vec{S}^{E_1;E_2;\ldots;E_K}$. In particular, we know that $dim(TS_{E_i}^{E_1;E_2;\ldots;E_K}) \geq 1$ for each i. Further assume, without loss of generality, that $|E_1| > 1$ (since $K < n$ one such clique must exist). Choose $v_1 = \lambda_{E_2} - \lambda_{E_1}$, so $v_1 \notin NS_{E_1}^{E_1;E_2;\ldots;E_K}(\vec{\lambda})$.

Consider some set of sufficiently small ϵ_i for $i = 1, \ldots, |E_1|$ and then consider the profile $\vec{\gamma}$ where $\gamma_i = \lambda_i$ for $i \notin E_1$ and $\gamma_i = \lambda_i + \epsilon_i v_1$ for $i \in E_1$. For sufficiently small ϵ_i, $D(\vec{\gamma}) \geq K$ and $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\gamma})\| < \delta$, and we have assumed that $D \leq K$ for all such $\vec{\gamma}$, so we must have $D(\vec{\gamma}) = K$.

Now, consider any two elements with $\epsilon_i = \epsilon_j$ to be elements of the same clique and then apply the reduction process to the profile $\vec{\gamma}$. None of the elements outside of E_1 are modified by the reduction process: $R_i(\vec{\gamma}) = \gamma_i$ for all $i \notin E_1$. The only possible modifications would be to merge some of the various elements of E_1 together into cliques. The resulting profile $\vec{R}(\vec{\gamma})$ must be in $\mathcal{Y}(S^{e_1;\ldots;e_s;E_2;\ldots;E_K})$ for some set of disjoint cliques $e_1, \ldots, e_s$, with $\cup_{a=1}^{s} e_a = E_1$. Assume that there are at least two such cliques e_a (i.e., $s > 1$), and also assume that the last merging step is to combine two disjoint cliques $\tilde{e}_a$ and $\tilde{e}_b$ to create e_1: $e_1 = \tilde{e}_a \cup \tilde{e}_b$. Let $\gamma_{\tilde{e}_a}$ and $\gamma_{\tilde{e}_b}$ denote the slope of the utility functions of those two cliques before merging. The cliques $\tilde{e}_a$ and $\tilde{e}_b$ must have been either fully or partially coalesced before being merged. However, if they were fully coalesced then the profiles $(\vec{R}(\vec{\gamma})|^{e_1}\gamma_{\tilde{e}_a})$ and $(\vec{R}(\vec{\gamma})|^{e_1}\gamma_{\tilde{e}_b})$ must both be in $\mathcal{Y}(S^{e_1;\ldots;e_s;E_2;\ldots;E_K})$ and produce identical allocations. Thus, $\vec{S}(\vec{R}(\vec{\gamma})|^{e_1}\lambda_1 + yv_1)$ is independent of y; in particular, this must hold for $y = 0$ and $y = 1$ which is clearly untrue since $\lambda_1 + v_1 = \lambda_2$. Similarly, if the last two cliques $\tilde{e}_a$ and $\tilde{e}_b$ to be merged were only partially coalesced then the profile $(\vec{R}(\vec{\gamma})|^{\tilde{e}_a;\tilde{e}_b}\gamma_{\tilde{e}_a}, \gamma_{\tilde{e}_b})$ before this last coalescence is in $\mathcal{Z}(\vec{S}^{\tilde{e}_a;\tilde{e}_b;e_2;\ldots;e_s;E_2;E_3;\ldots;E_K})$ and so we can invoke the smoothness assumptions which imply that either setting $\gamma_{\tilde{e}_a} = \lambda_2$ or setting $\gamma_{\tilde{e}_b} = \lambda_2$ does not change the allocation, which clearly cannot hold. Thus, there can be no last merging, and so the reduction process leaves $\vec{\gamma}$ unchanged. Therefore, the allocations of any pair of elements $i, j \in E_1$ are distinct if and only if $\epsilon_i \neq \epsilon_j$. If we choose all of the ϵ_i's to be distinct then $D(\vec{\gamma}) = K + |E_1| - 1 > K$ which contradicts our assumption. This completes the proof of the claim.$\diamond$

For future reference, note that the coalescing order of the elements in E_1 must, at this new profile $\vec{\gamma}$, be either in increasing or decreasing order of the ϵ_i. If not, then one could find at least two different coalescing orders; the tailmost element in E_1, call it i, could either be increased to coalesce with some element j with an adjacent (i.e., there are no other ϵ_k's in between) but larger value for ϵ_j, or it could equally well be decreased to coalesce with some element j with an adjacent but smaller value for ϵ_j.

We now address the case where $D(\vec{\lambda}) = 1$.

Claim: For any $\vec{\lambda} \in \mathcal{X}(\vec{S})$, if $1 = D(\vec{\lambda})$ then for any $\delta > 0$ there exists some $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta$ and $D(\vec{\eta}) > D(\vec{\lambda})$.

Proof: Assume, to the contrary, that there is some $\vec{\lambda} \in \mathcal{X}(\vec{S})$ and $\delta > 0$ such that $D(\vec{\lambda}) = 1$ and $D(\vec{\eta}) = 1$ for all $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta$. In this case, at every such profile $\vec{\eta}$ we have $S_i(\vec{\eta}) = S_j(\vec{\eta})$ for all i, j. After doing the reduction on such a profile $\vec{\eta}$, there is always only one remaining clique so $R_i(\vec{\eta}) = R_j(\vec{\eta})$ for all i, j. Furthermore, for every such profile $\vec{\eta}$ there is some $I(\vec{\eta})$ such that $\vec{S}(\vec{\eta}) = \vec{S}(\vec{\eta}|^N\eta_j)$ and $\vec{R}(\vec{\eta}) = (\vec{\eta}|^N\eta_j)$ for $j = I(\vec{\eta})$.

We now show that we must have $dim(TS_N^N(\vec{R}(\vec{\eta}))) \geq 1$ for all such $\vec{\eta}$. To see this, assume to the contrary that $dim(TS_N^N(\vec{R}(\vec{\eta}))) = 0$ for at least one $\vec{\eta}$. Then, S_N^N is constant; i.e., $S_N^N(\vec{\lambda}|^N\gamma)$ is independent of γ and $dim(TS_N^N(\vec{\lambda}|^N\gamma)) = 0$ for all γ. If there is any $\vec{\lambda} \in \mathcal{X}(\vec{S})$ such that $D(\vec{\lambda}) > 1$ then at this point we know that $dim(TS_N^N(\vec{\lambda}|^N\lambda_j)) \geq 1$ if j is the last element in the coalescing hierarchy (farthest from the tail). If $D(\vec{\lambda}) = 1$ for all $\vec{\lambda} \in \mathcal{X}(\vec{S})$ then, because $\vec{S}$ is nonconstant, we must have $dim(TS_N^N(\vec{R}(\vec{\lambda}))) \geq 1$ for all $\vec{\lambda} \in \mathcal{X}(\vec{S})$. Thus, we must have $dim(TS_N^N(\vec{R}(\vec{\eta}))) \geq 1$ for all $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta$.

Consider some $\vec{\eta} \in \mathcal{X}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\eta})\| < \delta/2$, and then choose some $v \in TS_N^N(\vec{\eta}|^N \eta_j)$ with $j = I(\vec{\eta})$. Consider a sufficiently small $\vec{\epsilon}$ with all ϵ_i distinct, and then consider a profile $\vec{\gamma}(\vec{\epsilon})$ of the form $\gamma_i(\vec{\epsilon}) = \eta_j + \epsilon_i v$ with $j = I(\vec{\eta})$. Note that for sufficiently small ϵ_i, $v \notin NS_N^N(\vec{\eta}|^N \eta_j + \epsilon_i v)$ for any i, again with $j = I(\vec{\eta})$. Thus, $\vec{S}(\vec{\eta}|^N \gamma_i(\vec{\epsilon})) \neq \vec{S}(\vec{\eta}|^N \gamma_k(\vec{\epsilon}))$ for $i \neq k$ thereby showing that the element $I(\vec{\gamma}(\vec{\epsilon}))$ is not fully coalesced with any other element. Since $\vec{S}$ is continuous, $I(\vec{\gamma}(\vec{\epsilon}))$ will not change as we continuously vary $\vec{\epsilon}$ as long as we preserve the inequality $\epsilon_k \neq \epsilon_j$ for all $k \neq j$, with $j = I(\vec{\gamma}(\vec{\epsilon}))$. This leads to two contradictory results.

First, note that $I(\vec{\gamma}(\vec{\epsilon})) = I(\vec{\gamma}(\vec{\epsilon}'))$ for all sufficiently small $\vec{\epsilon}'$ such that all ϵ_i' are distinct and the order of the elements in terms of increasing magnitude in $\vec{\epsilon}'$ is the same as in $\vec{\epsilon}$. By continuity and anonymity, then, for all sufficiently small $\vec{\epsilon}'$ with elements labelled in nondecreasing order, $I(\vec{\gamma}(\vec{\epsilon}'))$ is independent of $\vec{\epsilon}'$.

Second, by continuously varying the ϵ_i we can form cliques (perhaps empty) e_a and e_b of elements with $\epsilon_a < \epsilon_j$ and $\epsilon_b > \epsilon_j$ with $j = I(\vec{\gamma}(\vec{\epsilon}))$. Furthermore, the allocation remains unchanged when we merge these two cliques e_a and e_b into a single clique $e_c \equiv N - I(\vec{\gamma}(\vec{\epsilon}))$ because they are fully coalesced (if we assume that they aren't fully coalesced then it follows immediately that the profile is in $\mathcal{Z}(\vec{S}^{e_a;e_b;I})$ and then v must be in $NS_{e_a}^{e_a;e_b;I}(\vec{\gamma}(\vec{\epsilon}))$ and $NS_{e_b}^{e_a;e_b;I}(\vec{\gamma}(\vec{\epsilon}))$, which then implies that they are fully coalesced). Because this single clique is not fully coalesced with the element $I(\vec{\gamma}(\vec{\epsilon}))$, we can vary the ϵ_c associated with this clique to any value without changing the allocation: $\vec{S}(\vec{\gamma}(\vec{\epsilon}|^{e_c}\kappa))$ is independent of κ. By considering two sufficiently small values of κ, one that is larger than ϵ_j and one that is smaller than ϵ_j (with $j = I(\vec{\gamma}(\vec{\epsilon}))$), one can see that this contradicts the first result that, for all sufficiently small $\vec{\epsilon}'$ with elements labelled in nondecreasing order, $I(\vec{\gamma}(\vec{\epsilon}'))$ is independent of $\vec{\epsilon}'$. This establishes our claim.$\diamond$

By repeatedly applying these claims until we find a profile $\vec{\gamma} \in \mathcal{X}(\vec{S})$ with $D(\vec{\gamma}) = n$, we can conclude that for any $\vec{\lambda} \in \mathcal{X}(\vec{S})$ and any $\delta > 0$ we can construct a new profile $\vec{\gamma} \in \mathcal{Y}(\vec{S})$ such that $\|\vec{S}(\vec{\lambda}) - \vec{S}(\vec{\gamma})\| < \delta$. Thus, $\mathcal{Y}(\vec{S})$ is nonempty and $\vec{S}(\mathcal{X}(\vec{S})) \in \overline{\vec{S}(\mathcal{Y}(\vec{S}))}$.

Step 7 *Miscellaneous Results*

We now establish some minor results that we will use in subsequent steps. Consider some $k \in T(2)$ and some profile $(\vec{\lambda}|^{T(k)}\lambda_{k-1}) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;T(k-1)})$. If, for a given $v \in \Re^m$ and for all sufficiently small $\epsilon > 0$, $(\vec{\lambda}|^{T(k)}\lambda_{k-1} + \epsilon v) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;T(k)})$ and element $T(k)$ is the tail element, then for all sufficiently small $\epsilon > 0$ $(\vec{\lambda}|^{T(k)}\lambda_{k-1} - \epsilon v) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;T(k)})$ and element $k-1$ is the tail element. To see this, consider some $\epsilon > 0$ and some set of sufficiently small ϵ_i, $i \in T(k-1)$, such that $\epsilon_i < \epsilon_{i+1}$ for $k-1 \leq i < n$. Construct the profile $\vec{\gamma} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_{k-2}, \lambda_{k-1} + \epsilon_{k-1}v, \lambda_{k-1} + \epsilon_k v, \ldots, \lambda_{k-1} + \epsilon_n v)$. We know that $v \notin NS_{T(k-1)}^{T(k-1)}$ and so $\vec{\gamma} \in \mathcal{Y}(\vec{S})$ and element n is the tail element. By varying the ϵ_i appropriately, one can approach arbitrarily closely to permutations of both of the profiles $(\vec{\lambda}|^{T(k)}\lambda_{k-1} + \epsilon v)$ and $(\vec{\lambda}|^{T(k)}\lambda_{k-1} - \epsilon v)$ while still maintaining the inequalities $\epsilon_i < \epsilon_{i+1}$ for $k \leq i < n$. As we vary the ϵ_i, the tail element (or clique) must be the element (or clique) with the largest value of ϵ_i (recall the observation at the end of the construction used to prove the first claim in

Step 6). Thus, for all sufficiently small $\epsilon > 0$ $(\vec{\lambda}|^{T(k)}\lambda_{k-1} - \epsilon v) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;T(k)})$ and element $k-1$ is the tail element.

We have shown that for any profile $\vec{\lambda} \in \mathcal{X}(\vec{S})$ with $\vec{R}(\vec{\lambda}) = \vec{\lambda}$ there is a set of cliques E_i such that $\vec{\lambda} \in \mathcal{Y}(\vec{S}^{E_1;E_2;\ldots;E_n})$. With the labelling so that element E_1 is at the bottom of the coalescing order (farthest from the tail), let v denote a unit vector in $TS_{E_1}^{E_1;E_2;\ldots;E_n}(\vec{\lambda})$. Then, for small enough $\epsilon > 0$, the profile $\vec{\gamma}$ with $\gamma_i = \lambda_{E_1} + i\epsilon v$ will be in $\mathcal{Y}(\vec{S})$. Either element n or element 1 is the tail element. We say that v *points towards the tail* if element n is the tail element. Consider any $\vec{\lambda} \in \mathcal{Y}(\vec{S})$ labelled according to the coalescing order. For any $1 < k < n$, consider any $v \in TS_{T(k)}^{1;2;\ldots;T(k)}(\vec{\lambda}|^{T(k)}\lambda_k)$ which points toward the tail. Then, for all sufficiently small $\epsilon > 0$, $(\vec{\lambda}|^{T(k+1)}\lambda_k + \epsilon v) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;k;T(k+1)})$ and element $T(k+1)$ is the tail element for all sufficiently small $\epsilon > 0$. Conversely, if for some γ we have $(\vec{\lambda}|^{T(k+1)}\gamma) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;k;T(k+1)})$ and element $T(k+1)$ is the tail element then $\gamma - \lambda_k = v + w$ for some $v \in TS_{T(k)}^{1;2;\ldots;T(k)}(\vec{\lambda}|^{T(k)}\lambda_k)$ which points toward the tail and some $w \in NS_{T(k)}^{1;2;\ldots;T(k)}(\vec{\lambda}|^{T(k)}\lambda_k)$.

Similarly, at any $\vec{\lambda} \in \mathcal{Y}(\vec{S})$ labelled according to the coalescing order, and for any $v \in TS_i(\vec{\lambda})$, we say that v points toward the tail if $v \cdot \lambda_{i-1} > 0$ or $v \cdot \lambda_{i+1} < 0$. Note that this is well defined, since we cannot have $v \cdot \lambda_{i-1} > 0$ and $v \cdot \lambda_{i+1} > 0$ (as the previous paragraph shows).

Step 8 *$dim(TS_i(\vec{\lambda})) = 1$ for all $i < n$.*

Assume, to the contrary, that $dim(TS_i(\vec{\lambda})) > 1$ for some $i < n$. For convenience, denote $(\lambda_{i+1} - \lambda_i)$ by Δ_i. Choose some $v \in TS_i(\vec{\lambda}) = TS_{T(i)}^{1;2;\ldots;T(i)}(\vec{\lambda}|^{T(i)}\lambda_i)$ such that $v \cdot \Delta_i > 0$ and $w \cdot \Delta_i = 0$ for all $w \in TS_i(\vec{\lambda})$ such that $v \cdot w = 0$. Since $dim(TS_i(\vec{\lambda})) > 1$, we can choose some $w \in TS_i(\vec{\lambda})$ such that $v \cdot w = 0$. Consider, for some sufficiently small fixed $\epsilon > 0$ and a variable $0 \le \theta \le 2\pi$, the profile $\vec{\gamma}(\theta) = (\vec{\lambda}|^{T(i+1)}\lambda_i + \epsilon \sin\theta w + \epsilon\cos\theta\Delta_i)$.

Clearly for $\theta = 0$ we have $S_{T(i+1)}^{1;2;\ldots;i;T(i+1)}(\vec{\gamma}(0)) \ne S_i^{1;2;\ldots;i;T(i+1)}(\vec{\gamma}(0))$ and $\vec{\gamma}(0) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;i;T(i+1)})$. Note that for all θ, $\vec{\gamma}(\theta) \in \mathcal{Z}(\vec{S}^{1;2;\ldots;i;T(i+1)})$ since the two elements i and $T(i+1)$ cannot fully coalesce since $(\epsilon\sin\theta w + \epsilon\cos\theta\Delta) \notin NS_{T(i)}^{1;2;\ldots;T(i)}(\vec{\lambda}|^{T(i)}\lambda_i)$. Let $\bar{\theta}$ denote, if it exists, the minimal value of $\theta \ge 0$ such that there is some other tail element besides element $T(i+1)$. For sufficiently small ϵ, $\vec{\gamma}(\theta) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;i;T(i+1)})$ for all $\theta \in [0, \bar{\theta})$. Element i must be the additional tail element at the profile $\vec{\gamma(\bar{\theta})}$ since element $T(i+1)$ cannot coalesce with any other element if we choose ϵ sufficiently small. However, as we observed at the end of Step 3, two tail elements cannot partially coalesce unless the differences in their λ's are in both null spaces and we know that $(\epsilon\sin\theta w + \epsilon\cos\theta\Delta) \notin NS_{T(i)}^{1;2;\ldots;T(i)}(\vec{\lambda}|^{T(i)}\lambda_i)$. Thus, there can be no such $\bar{\epsilon}$ and so $\vec{\gamma}(\theta) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;i;T(i+1)})$ for all θ (with element $T(i+1)$ remaining the unique tail element). In particular, this applies to $\theta = \pi$, so element $T(i+1)$ is the unique tail element at the profile $(\vec{\lambda}|^{T(i+1)}\lambda_i - \epsilon\Delta)$.

The results of Step 7 imply that if for all sufficiently small $\epsilon > 0$ element $T(i+1)$ is the unique tail at the profile $(\vec{\lambda}|^{T(i+1)}\lambda_i + \epsilon\Delta)$ then for all sufficiently small $\epsilon > 0$ element i is the unique tail at the profile $(\vec{\lambda}|^{T(i+1)}\lambda_i - \epsilon\Delta)$. This contradicts the assertion in the previous paragraph that element $T(i+1)$ is a tail element at the profile $(\vec{\lambda}|^{T(i+1)}\lambda_i - \epsilon\Delta)$. Therefore, we must have $dim(TS_i(\vec{\lambda})) = 1$.

Step 9 $dim(TS_n(\vec{\lambda})) = 1$

Let w denote the unique vector such that (1) $\|w\| = 1$, (2) $w \in TS_{n-1}(\vec{\lambda})$, and (3) $w \cdot (\lambda_n - \lambda_{n-1}) > 0$. For any $v \in NS_{n-1}(\vec{\lambda})$ we have $\vec{S}(\vec{\lambda}|^n \lambda_{n-1} + \epsilon v) = \vec{S}(\vec{\lambda}|^n \lambda_{n-1})$ for all ϵ such that $\lambda_{n-1} + \epsilon v \in \Re^m_+$.

Define $H : \Re^m_+ \mapsto \Re^m_{++}$ by $H^\alpha(x) = \frac{\partial A^\nu}{\partial x^\alpha}(x)$ for some $\nu \in P$. Define $\mathcal{T} : \Re^m_{++} \mapsto \Re^m_+$ by $\mathcal{T}(\vec{\lambda}) = \sum_{i=1}^n S_i(\vec{\lambda})$. Recall that $\vec{S}(\vec{\lambda}|^n H(\mathcal{T}(\vec{\lambda}))) = \vec{S}(\vec{\lambda})$, since $H(\mathcal{T}(\vec{\lambda})) \in NS_n(\vec{\lambda})$. Define the two matrices $C_{\alpha,\beta} \equiv \frac{\partial S_n^\alpha}{\partial \lambda_n^\beta}(\vec{\lambda})$ and $D_{\alpha,\beta} \equiv \frac{\partial H^\alpha}{\partial x^\beta} = \frac{\partial^2 A^\nu}{\partial x^\alpha \partial x^\beta}$; note that $D_{\alpha,\beta}$ is symmetric, and is thus diagonalizable. Furthermore, the norm of $D_{\alpha,\beta} = \frac{\partial^2 A^\nu}{\partial x^\alpha \partial x^\beta}$, when evaluated at any point in $\mathcal{T}(\mathcal{L}^n)$, is bounded above by some $\Omega > 0$ (i.e., $\frac{\|Dw\|}{\|w\|} < \Omega$ for all w). In other words, $\Omega > \frac{D^x w}{\|w\|}$ for all $w \in \Re^m$ and $x \in \mathcal{T}(\mathcal{L}^n)$ where D^x denotes the matrix D that results from evaluating the defining derivatives at the point x.

Taking derivatives of the relation $\vec{S}(\vec{\lambda}|^n H(\mathcal{T}(\vec{\lambda}))) = \vec{S}(\vec{\lambda})$, we find that $C_{\alpha,\beta} = \sum_{\gamma,\mu} C_{\alpha,\gamma} D_{\gamma,\mu} C_{\mu,\beta}$. Or, using notation that denotes matrix multiplication, $C = CDC$. Since $TS_n(\vec{\lambda})$ is the range of C and $NS_n(\vec{\lambda})$ is the null space of C, we know that CD is the identity on $TS_n(\vec{\lambda})$. Since the norm of all such matrices D is bounded above by Ω, then for any $w \in TS_n(\vec{\lambda})$, $\frac{\|Cw\|}{\|w\|} > \frac{1}{\Omega}$. Furthermore, the bound on the second derivatives $\frac{\partial^2 S_k^\gamma}{\partial \lambda_i^\alpha \partial \lambda_j^\beta}(\vec{\lambda})$, which is ensured by the smoothness conditions, implies that there exists some $\Pi > 0$ and $\pi > 0$ such that $\|\vec{S}(\vec{\lambda}|^n \lambda_n + \pi v) - \vec{S}(\vec{\lambda})\| > \Pi$ for all $v \in TS_n(\vec{\lambda})$ with $\|v\| = 1$. These bounds with fixed constants π, Π apply to all $\vec{\lambda} \in \mathcal{Y}(S)$ with element n being the tail, and to all $v \in TS_n(\vec{\lambda})$ with $\|v\| = 1$.

Assume that $dim(TS_n(\vec{\lambda})) > 1$. Then, there is some $\bar{\epsilon} \in (0,1)$ such that $dim(TS_n(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n)) > 1$ for all $\epsilon \in [0, \bar{\epsilon})$. If $dim(TS_n(\vec{\lambda}|^n \lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n)) = 1$ then for each $\epsilon \in [0, \bar{\epsilon})$ we can find some continuous function $w(\epsilon) \in NS_n(\vec{\lambda}|^n \lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n))$ such that $w(\epsilon) \in TS_n(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n))$ and $\|w(\epsilon)\| = 1$. For this function $w(\epsilon)$ we have $\lim_{\epsilon \to \bar{\epsilon}^-} \|\vec{S}(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n) + \pi w(\epsilon)) - \vec{S}(\vec{\lambda} + \epsilon(\lambda_n - \lambda_{n-1}))\| \geq \Pi \neq 0$. This contradicts the fact that for all such $w(\epsilon)$ we have $\vec{S}(\vec{\lambda}|^n \lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n) + \pi w(\bar{\epsilon})) = \vec{S}(\vec{\lambda}|^n \lambda_n + \bar{\epsilon}(\lambda_{n-1} - \lambda_n))$. Thus, $dim(TS_n(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n)) > 1$ for all $\epsilon \in [0,1)$. Furthermore, we can find some continuous function $w(\epsilon) \in NS_{n-1}(\vec{\lambda})$ such that $w(\epsilon) \in TS_n(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n))$ and $\|w(\epsilon)\| = 1$. For this function $w(\epsilon)$ we have $\lim_{\epsilon \to \bar{\epsilon}^-} \|\vec{S}(\vec{\lambda}|^n \lambda_n + \epsilon(\lambda_{n-1} - \lambda_n) + \pi w(\epsilon)) - \vec{S}(\vec{\lambda} + \epsilon(\lambda_n - \lambda_{n-1}))\| \geq \Pi \neq 0$. This contradicts the fact that for all such $w(\epsilon)$ we have $\vec{S}(\vec{\lambda}|^n \lambda_{n-1} + \pi w(1)) = \vec{S}(\vec{\lambda}|^n \lambda_{n-1})$. Contradiction. Therefore, $dim(TS_n(\vec{\lambda})) = 1$.

Step 10 *Relation to generalized serial mechanisms*

Recall that in Step 6 we showed that $\vec{S}(\mathcal{X}(\vec{S})) \subseteq \overline{\vec{S}(\mathcal{Y}(\vec{S}))}$. Furthermore, the smoothness condition required that $\vec{S}(\mathcal{L}^n) \subseteq \overline{\vec{S}(\mathcal{X}(\vec{S}))}$. Therefore, $\vec{S}(\mathcal{L}^n) \subseteq \overline{\vec{S}(\mathcal{Y}(\vec{S}))}$.

Consider, for any $\vec{\lambda} \in \mathcal{Y}(\vec{S})$ with the elements labelled according to the coalescing order and for any value of k with $1 \leq k \leq n$, the sets

$$B^k(\vec{\lambda}) = \{S_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\gamma) | (\vec{\lambda}|^{T(k)}\gamma) \in \mathcal{Y}(\vec{S}^{1;\ldots;T(k)}),\ and\ T(k)\ is\ the\ tail\}$$

$$C^k(\vec{\lambda}) = \{\gamma | (\vec{\lambda}|^{T(k)}\gamma) \in \mathcal{Y}(\vec{S}^{1;\ldots;T(k)}),\ \textit{and } T(k) \textit{ is the tail}\}$$

Obviously, $B^k(\vec{\lambda}) = S_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}C^k(\vec{\lambda}))$. In addition, for any $v \in TS_{k-1}^{1;\ldots;k-2;T(k-1)}(\vec{\lambda}|^{T(k-1)}\lambda_{k-1})$ which points toward the tail, and any $\gamma \in C^k(\vec{\lambda})$, we have $v \cdot \gamma > 0$ as was shown in Step 7.

Claim: $B^k(\vec{\lambda})$ and $C^k(\vec{\lambda})$ are connected.

Proof: Consider two points γ and η such that $\gamma \in C^k(\vec{\lambda})$ and $\eta \in C^k(\vec{\lambda})$, and then construct the path $(\vec{\lambda}|^{T(k)}\gamma + \epsilon(\eta-\gamma))$ for $\epsilon \in [0,1]$. Clearly, $(\vec{\lambda}|^{T(k)}\gamma + \epsilon(\eta-\gamma)) \in \mathcal{X}(\vec{S})$ for all $\epsilon \in [0,1]$, since the smoothness condition (condition 4 in the definition of Z-smoothness) requires that if the profile leaves $\mathcal{X}(\vec{S})$ along such a linear path it cannot return. Let v denote an element in $TS_{k-1}^{1;2;\ldots;T(k-1)}(\vec{\lambda}|^{T(k)}\lambda_{k-1})$ which points toward the tail; then $v\cdot(\lambda_{k-1}-\gamma) > 0$ and $v\cdot(\lambda_{k-1}-\eta) > 0$. As we increase ϵ from $\epsilon = 0$ to $\epsilon = 1$, the quantity $v \cdot (\lambda_{k-1} - \gamma + \epsilon(\eta-\gamma)) > 0$ always. Thus, element $T(k)$ cannot coalesce with element $k-1$ anywhere on the path. Thus, $(\vec{\lambda}|^{T(k)}\gamma+\epsilon(\eta-\gamma)) \in \mathcal{Y}(\vec{S}^{1;2;\ldots;T(k)})$ for all $\epsilon \in [0,1]$ and element $T(k)$ remains the tail. Thus, the set $C^k(\vec{\lambda})$ is connected. Continuity of $\vec{S}$ then implies that $B^k(\vec{\lambda})$ is connected.$\diamond$

Define $g(\lambda) = S_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\lambda)$. For all $\lambda \in \{\gamma \in \Re_+ | \exists \eta \in C^k(\vec{\lambda})\ \textit{such that}\ g(\eta) = g(\gamma)\}$, define $v(\lambda)$ to be, for any $\eta \in C^k(\vec{\lambda})$ *such that* $g(\eta) = g(\lambda)$, the unique member of $TS_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\eta)$ that has unit length and points toward the tail. Note that $v(\lambda)\cdot\lambda = 0$. In the following series of claims, assume that $\gamma, \eta, \xi \in C^k(\vec{\lambda})$ unless specified otherwise.

Claim: $v(\gamma)\cdot\eta = 0 \Leftrightarrow v(\eta) = v(\gamma)$.

Proof: If $v(\eta) = v(\gamma)$ then $v(\gamma)\cdot\eta = 0$ because $v(\lambda)\cdot\lambda = 0$ for all $\lambda \in C^k(\vec{\lambda})$. If $v(\gamma)\cdot\eta = 0$ then $g(\gamma+s(\eta-\gamma))$ is constant so $TS_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\gamma) = TS_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\eta)$ and thus $v(\eta) = v(\gamma)$.$\diamond$

Claim: $v(\gamma)\cdot\eta > 0 \Leftrightarrow v(\eta)\cdot\gamma < 0$.

Proof: If $(\eta-\gamma)\cdot v(\gamma+s(\eta-\gamma)) = 0$ for any s, then $g(\gamma+s(\eta-\gamma))$ is constant, so $v(\eta) = v(\gamma)$ and $v(\gamma)\cdot\eta = 0$. Thus, if $v(\gamma)\cdot\eta > 0$ then $(\eta-\gamma)\cdot v(\gamma+s(\eta-\gamma)) > 0$ for all $s \in [0,1]$. In particular, $0 < (\eta-\gamma)\cdot v(\eta) = -v(\eta)\cdot\gamma$. The converse follows from symmetry.$\diamond$

Define $f(s) = g(\gamma+s(\eta-\gamma))$ for $s \in [0,1]$. If γ and η are such that $f(s)$ is not constant, then, $f'(0) = c_0 v(\gamma)$ and $f'(1) = c_1 v(\eta)$ for some nonzero constants c_0, c_1; furthermore, since smoothness demands that $f'(s)\cdot(\eta-\gamma) > 0$, we then know that $c_0 c_1 > 0$.

Claim: $(\exists \bar{s} \in [0,1]\ \textit{such that}\ f(\bar{s}) = g(\xi)) \Leftrightarrow (v(\gamma)\cdot\xi)(v(\eta)\cdot\xi) \leq 0$.

Proof: If $f(s)$ is constant then teh conclusion follows immediately since the righthand side is satisfied only if $(v(\gamma)\cdot\xi) = 0$ which then implies $g(\xi) = g(\gamma)$. If $f(s)$ is not constant, then define $z(s) = f'(s)\cdot[\gamma+s(\eta-\gamma)-\xi]$. Note that $f(s) = g(\xi) \Leftrightarrow z(s) = 0$ because $(\gamma + s(\eta-\gamma)-\xi) \in NS_{T(k)}^{1;\ldots;k-1;T(k)}(\vec{\lambda}|^{T(k)}\gamma+s(\eta-\gamma)) \Leftrightarrow f'(s)\cdot(\gamma+s(\eta-\gamma)-\xi) = 0$. Note that $z(0) = -\xi\cdot f'(0) = -c_0 v(\gamma)\cdot\xi$ and $z(1) = -\xi\cdot f'(1) = -c_1 v(\eta)\cdot\xi$. If $(v(\gamma)\cdot\xi)(v(\eta)\cdot\xi) \leq 0$ then $z(0)z(1) \leq 0$ and so there is some $\bar{s} \in [0,1]$ such that $z(\bar{s}) = 0$; therefore, $\exists \bar{s} \in [0,1] \ni f(\bar{s}) = g(\xi)$. If $(v(\gamma)\cdot\xi)(v(\eta)\cdot\xi) > 0$ then $(v(\xi)\cdot\gamma)(v(\xi)\cdot\eta) > 0$ so for any $s \in [0,1]$ $v(\xi)\cdot(\gamma+s(\eta-\gamma)) \neq 0$ implying that $g(\xi) \neq f(s)$ for any $s \in [0,1]$.$\diamond$

Claim: $(v(\gamma)\cdot\eta > 0 \; and \; v(\eta)\cdot\xi > 0) \Rightarrow v(\gamma)\cdot\xi > 0$

Proof: Assume, to the contrary, that $v(\gamma)\cdot\eta > 0 \; and \; v(\eta)\cdot\xi > 0$ but $v(\xi)\cdot\gamma > 0$. Consider the family $\gamma + s(\eta-\gamma)$. From above, we know that for every $s \in (0,1)$ we must have $v(\gamma + s(\eta - \gamma))\cdot(\eta-\gamma) > 0$. In addition, there must be some $\bar{s} \in (0,1)$ such that $g(\gamma+\bar{s}(\eta-\gamma)) = g(\xi)$ since $(v(\gamma)\cdot\xi)(v(\eta)\cdot\xi) < 0$. Therefore, $v(\xi)\cdot(\eta-\gamma) = v(\gamma+\bar{s}(\eta-\gamma))\cdot(\eta-\gamma) > 0$. However, by direct computation, $v(\xi)\cdot(\eta-\gamma) < 0$ since each term is assumed to be negative. Contradiction.$\diamond$

Thus, one can set up a transitive ordering among elements of the set $C^k(\vec{\lambda})$ by considering $\gamma > \eta$ if $v(\eta)\cdot\gamma > 0$. The ordering on $C^k(\vec{\lambda})$ induces an analogous ordering on $B^k(\vec{\lambda})$, and also on the set $\{\gamma \in \Re^m_+ | g(\gamma) \in B^k(\vec{\lambda})\}$. Define the function $e : M \mapsto \Re^m_+$ by $e^\beta(\alpha) = 0$ if $\beta \neq \alpha$ and $e^\beta(\alpha) = 1$ if $\beta = \alpha$. Let $\mathcal{B} = \{\alpha \in M | g(e(\alpha)) \in B^k(\vec{\lambda})\}$, and note that $\alpha \in \mathcal{B} \Leftrightarrow (\exists \gamma \in C^k(\vec{\lambda}) \; such \; that \; v(\gamma)\cdot e(\alpha) = 0)$. Let $\mathcal{B}_t = \{\alpha \in M | e(\alpha)\cdot v(\gamma) > 0 \; for \; all \; \gamma \in C^k(\vec{\lambda})\}$ and $\mathcal{B}_b = \{\alpha \in M | e(\alpha)\cdot v(\gamma) < 0 \; for \; all \; \gamma \in C^k(\vec{\lambda})\}$. The three sets $\mathcal{B}$, $\mathcal{B}_t$, and $\mathcal{B}_b$ are a partition of M.

Claim: $\mathcal{B}_t$ and $\mathcal{B}_b$ are both nonempty.

Proof: Assume that $\mathcal{B}_b$ is empty. Then, at any $\gamma \in C^k(\vec{\lambda})$ such that $v(\gamma)\cdot e(\alpha) \geq 0$ for all α (such γ's must exist), we have $v(\gamma) \in \Re^m_+$ which violates strategy-proofness. A similar argument applies if we assume $\mathcal{B}_t$ is empty.$\diamond$

Claim: For any $\alpha_b \in \mathcal{B}_b$, $\alpha_t \in \mathcal{B}_t$, there exists at least one value of s such that $g(e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b))) \in B^k(\vec{\lambda})$.

Proof: Consider any $\gamma \in C^k(\vec{\lambda})$. Recall that we must have $e(\alpha_b)\cdot v(\gamma) < 0$ and $e(\alpha_t)\cdot v(\gamma) > 0$. Therefore, there is some s such that $(e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b)))\cdot v(\gamma) = 0$ and so $g(e(\alpha_1) + s(e(\alpha_2) - e(\alpha_1))) = g(\gamma) \in B^k(\vec{\lambda})$.$\diamond$

Claim: There exist α_b, α_t, and $\bar{s}_b, \bar{s}_t$ such that $B^k(\vec{\lambda}) = \{g(e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b))) | s \in (\bar{s}_b, \bar{s}_t)\}$.

Proof: Choose any $\alpha_b \in \mathcal{B}_b$ and $\alpha_t \in \mathcal{B}_t$. Define $\gamma(s) = e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b))$. We know $g(\gamma(0)) \notin B^k(\vec{\lambda})$ and $g(\gamma(1)) \notin B^k(\vec{\lambda})$. However, there is some $\bar{s}$ such that $g(\gamma(\bar{s})) \in B^k(\vec{\lambda})$. Let $\bar{s}_b$ denote the point at which $g(\gamma(s))$ leaves $B^k(\vec{\lambda})$ as s is decreased from $\bar{s}$, and let $\bar{s}_t$ denote the point at which $g(\gamma(s))$ leaves $B^k(\vec{\lambda})$ as s is increased from $\bar{s}$. Assume there is some point $x \in B^k(\vec{\lambda})$ such that $x \notin \{g(\gamma(s)) | s \in (\bar{s}_b, \bar{s}_t)\}$, and let $\xi \in C^k(\vec{\lambda})$ be such that $g(\xi) = x$. Because $B^k(\vec{\lambda})$ is connected, we can choose for any sufficiently small δ, without loss of generality, the point x and the associated ξ so that $\|x - g(\gamma(\tilde{s}))\| < \delta$ and $\|\xi - \gamma(\tilde{s}) - w\| < \delta$ for some $\tilde{s} \in (\bar{s}_b, \bar{s}_t)$ and some w such that $w \cdot v(\gamma(\tilde{s})) = 0$. Since $\gamma(\bar{s}_b)\cdot v(\gamma(\tilde{s})) > 0$ and $\gamma(\bar{s}_t)\cdot v(\gamma(\tilde{s})) < 0$, and δ is sufficiently small, there must be some $\hat{s}$ near $\tilde{s}$ such that $v(\gamma(\hat{s}))\cdot\xi = 0$ and thus $x = g(\gamma(\hat{s}))$. Contradiction.$\diamond$

Thus, the sets $B^k(\vec{\lambda})$ always have exactly two boundary points $g(e(\alpha_b) + \bar{s}_b(e(\alpha_t) - e(\alpha_b)))$ and $g(e(\alpha_b) + \bar{s}_t(e(\alpha_t) - e(\alpha_b)))$. For $k = 1$, these boundary points are both on the boundary of $\Re^m_+$. For $k > 1$, $g(e(\alpha_b) + \bar{s}_b(e(\alpha_t) - e(\alpha_b))) = S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\lambda_{k-1})$ and $g(e(\alpha_b) + \bar{s}_t(e(\alpha_t) - e(\alpha_b))) \in \partial\Re^m_+$. Define $h(s) = (e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b)))$.

Claim: For any $\gamma, \eta \in \partial C^k(\vec{\lambda})$ such that $S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\gamma) = g(h(\bar{s}_b))$ and $S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\eta) = g(h(\bar{s}_t))$, we have $B^k(\vec{\lambda}) = \{S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\gamma + z(\eta - \gamma)) | z \in (0,1)\}$.

Proof: For sufficiently small $\epsilon > 0$, $g(h(\bar{s}_b + \epsilon)) \in B^k(\vec{\lambda})$ and $g(h(\bar{s}_t - \epsilon)) \in B^k(\vec{\lambda})$. Consider any $\tilde{\gamma}, \tilde{\eta} \in C^k(\vec{\lambda})$ such that $S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\tilde{\gamma}) = g(h(\bar{s}_b + \epsilon))$ and $S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\tilde{\eta}) = g(h(\bar{s}_t - \epsilon))$. For all $z \in [0,1]$, $(\tilde{\gamma} + z(\tilde{\eta} - \tilde{\gamma})) \in C^k(\vec{\lambda})$. For any $s \in (\bar{s}_b + \epsilon, \bar{s}_t - \epsilon)$ we know that $v(\tilde{\gamma}) \cdot h(s) = v(h(\bar{s}_b + \epsilon)) \cdot h(s) > 0$ and $v(\tilde{\eta}) \cdot h(s) = v(h(\bar{s}_t - \epsilon)) \cdot h(s) < 0$ so there is some point $\tilde{z} \in (0,1)$ such that $v(\tilde{\gamma} + \tilde{z}(\tilde{\eta} - \tilde{\gamma})) \cdot h(s) = 0$. Therefore, $\{g(\tilde{\gamma} + z(\tilde{\eta} - \tilde{\gamma}))|z \in [0,1]\} = \{g(h(s))|s \in [\bar{s}_b + \epsilon, \bar{s}_b - \epsilon]\}$. Thus, when we take the limit of $\epsilon \to 0^+$, we arrive at the desired conclusion.$\diamond$

Recall that $H^\alpha(x) \equiv \frac{\partial A^\nu}{\partial x^\alpha}(x)$ for some $\nu \in P$. Define $p(s) = \mathcal{T}(\vec{\lambda}|^{T(k)}e(\alpha_b) + s(e(\alpha_t) - e(\alpha_b)))$ for $s \in (\bar{s}_b, \bar{s}_t)$. Define $\gamma(k; \vec{\lambda}) = H(p(\bar{s}_b))$ and $\eta(k; \vec{\lambda}) = H(p(\bar{s}_t))$. Thus, $B^k(\vec{\lambda}) = \{S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\gamma(k;\vec{\lambda}) + z(\eta(k;\vec{\lambda}) - \gamma(k;\vec{\lambda})))|z \in (0,1)\}$. Note that all the expressions are independent of λ_j for $j \geq k$.

We can define a function $\vec{r} : \mathcal{L}^n \mapsto [0,1]^n$ as follows. First, consider any $\vec{\lambda} \in \mathcal{Y}(\vec{S})$, labelled in the coalescing order. Let $r_1(\vec{\lambda})$ be the unique value of s such that $S_1(\vec{\lambda}) = S^{T(1)}_{T(1)}(\vec{\lambda}|^{T(1)}\gamma(1;\vec{\lambda}) + s(\eta(1;\vec{\lambda}) - \gamma(1;\vec{\lambda})))$. Assume that $r_j(\vec{\lambda})$ has been defined for $j < k$, and let $r_k(\vec{\lambda})$ be the unique value of s such that $S_k(\vec{\lambda}) = S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}|^{T(k)}\gamma(k;\vec{\lambda}) + \frac{s - r_{k-1}(\vec{\lambda})}{1 - r_{k-1}(\vec{\lambda})}(\eta(k;\vec{\lambda}) - \gamma(k;\vec{\lambda})))$. We can extend this function $\vec{r}(\vec{\lambda})$ symmetrically to all $\vec{\lambda} \in \mathcal{Y}(\vec{S})$. Note that if $\vec{\lambda}, \vec{\eta} \in \mathcal{Y}(\vec{S})$ then $\vec{S}(\vec{\lambda}) = \vec{S}(\vec{\eta}) \Leftrightarrow \vec{r}(\vec{\lambda}) = \vec{r}(\vec{\eta})$. Using this property, we can continuously extend $\vec{r}(\vec{\lambda})$ to all $\vec{\lambda} \in \mathcal{L}^n$ since $\vec{S}(\mathcal{L}^n) \subseteq \overline{\vec{S}(\mathcal{Y}(\vec{S}))}$. There we also have the property that for all $\vec{\lambda}, \vec{\eta} \in \mathcal{L}^n$, $\vec{S}(\vec{\lambda}) = \vec{S}(\vec{\eta}) \Leftrightarrow \vec{r}(\vec{\lambda}) = \vec{r}(\vec{\eta})$. Note that $\vec{r}(\mathcal{Y}(\vec{S})) = \{\vec{s} \in (0,1)^n|(i \neq j) \Rightarrow (s_i \neq s_j)\}$. Also note that for any $\vec{\lambda} \in \mathcal{L}^n$ we can define an ordering which is an extension of the coalescing ordering on $\mathcal{Y}(\vec{S})$; this ordering is merely the order of increasing $r_i(\vec{\lambda})$'s. Labelling the elements so that $r_j(\vec{\lambda}) \leq r_{j+1}(\vec{\lambda})$, we have that $r_i(\vec{\lambda}|^{T(k)}\lambda_k) = r_i(\vec{\lambda})$ for all $i \leq k$.

We now define a function $\vec{a} : [0,1]^n \mapsto \mathcal{A}$, and a function $\vec{\eta} : [0,1]^n \mapsto \mathcal{L}^n$ as follows. Consider some $\vec{s}$ such that $0 < s_1 < \ldots < s_n < 1$. Choose some arbitrary $\vec{\lambda}^0 \in \mathcal{Y}(\vec{S})$; this entire construction is independent of $\vec{\lambda}^0$ but our notation requires a placeholder to start the inductive definition. Consider the set $B^1(\vec{\lambda}^0)$, which is independent of $\vec{\lambda}^0$. This set can be parametrized $B^1(\vec{\lambda}^0) = \{S^{T(1)}_{T(1)}(\vec{\lambda}^0|^{T(1)}\gamma(1;\vec{\lambda}^0) + s(\eta(1;\vec{\lambda}^0) - \gamma(1;\vec{\lambda}^0)))|s \in (0,1)\}$. Set $a_1(\vec{s}) = S^{T(1)}_{T(1)}(\vec{\lambda}^0|^{T(1)}\gamma(1;\vec{\lambda}^0) + s_1(\eta(1;\vec{\lambda}^0) - \gamma(1;\vec{\lambda}^0)))$. Set $\vec{\lambda}^1 = (\vec{\lambda}^0|^{T(1)}\gamma(1;\vec{\lambda}^0) + s_1(\eta(1;\vec{\lambda}^0) - \gamma(1;\vec{\lambda}^0)))$. Now assume that $\vec{\lambda}^j$ and $a_j(\vec{s})$ have been defined for $j < k$. Then, set $a_k(\vec{s}) = S^{1;\ldots;k-1;T(k)}_{T(k)}(\vec{\lambda}^{k-1}|^{T(k)}\gamma(k;\vec{\lambda}^{k-1}) + \frac{s_k - s_{k-1}}{1 - s_{k-1}}(\eta(k;\vec{\lambda}^{k-1}) - \gamma(k;\vec{\lambda}^{k-1})))$ and $\vec{\lambda}^k = (\vec{\lambda}^{k-1}|^{T(k)}\gamma(k;\vec{\lambda}^{k-1}) + s_k(\eta(k;\vec{\lambda}^{k-1}) - \gamma(k;\vec{\lambda}^{k-1})))$. At the end of the construction, set $\vec{\eta}(\vec{s}) = \vec{\lambda}^n$. We can extend these functions $\vec{a}(\vec{s})$ and $\vec{\eta}(\vec{s})$ symmetrically to be defined on $\{\vec{s} \in (0,1)^n|(i \neq j) \Rightarrow (s_i \neq s_j)\}$. The image of such vectors under $\vec{a}(\vec{s})$ maps out all of $\vec{S}(\mathcal{Y}(\vec{S}))$. We can then extend this function continuously to all $\vec{s} \in [0,1]^n$. We then have $\vec{S}(\mathcal{L}^n) = \overline{\vec{S}(\mathcal{Y}(\vec{S}))} = \{\vec{a}(\vec{s})|\vec{s} \in [0,1]^n\}$. Note that the function $\vec{a}(\vec{s})$ has the property that $s_i > s_j \Rightarrow \frac{\partial a_j}{\partial s_i} = 0$. In fact, if the elements are ordered so that $s_j \leq s_{j+1}$ then $a_i(\vec{s}|^{T(k)}s_k) = a_i(\vec{s})$ for all $i \leq k$. By construction, $\vec{a}(\vec{r}(\vec{\lambda})) = \vec{S}(\vec{\lambda})$ for all $\vec{\lambda} \in \mathcal{L}^n$. Thus, when the elements are labelled so that $r_j(\vec{\lambda}) \leq r_{j+1}(\vec{\lambda})$ the following holds for all $i \leq k$: (1) $r_i(\vec{\lambda}|^{T(k)}\lambda_k) = r_i(\vec{\lambda})$, (2) $a_i(\vec{r}(\vec{\lambda})) = a_i(\vec{r}(\vec{\lambda})|^{T(i)}r_i(\vec{\lambda}))$, and (3) $S_i(\vec{\lambda}) = S_i(\vec{\lambda}|^{T(i)}\lambda_i)$. This ordering on all $\mathcal{L}^n$, like our coalescing ordering on $\mathcal{Y}(\vec{S})$, does not change as $\vec{\lambda}$ changes continuously until $r_i(\vec{\lambda}) = r_j(\vec{\lambda})$ for some $i \neq j$, and then the ordering changes only by those two elements i, j exchanging places in the ordering (if the values of r_i and r_j cross).

Consider any $\vec{t} \in [0,1]^n$ with $t_n = 1$ and define $\vec{\gamma}(s) = (\vec{\eta}(\vec{t})|^n \eta_n(\vec{t}|^n s))$. Note that for all s, we have that $t_i \leq s \Rightarrow r_i(\vec{\gamma}(s)) = t_i$ and $t_i \geq s \Rightarrow r_i(\vec{\gamma}(s)) \geq s$. This follows by noting that as s decreases from $s = 1$, the only changes in the coalescing ordering (and hence the ordering of the r_i's) for elements ahead of s (farther away from the tail) in the ordering occurs when s passes through some t_i and at that point they exchange places in the ordering (the relative ordering of elements behind s (closer to the tail) in the ordering can change arbitrarily among themselves as s decreases, but they can never pass s in the ordering). Therefore, for such $\vec{t}$'s we have that $S_n(\vec{\eta}(\vec{t}|^n s)) = S_n(\vec{\eta}(\vec{t})|^n \eta_n(\vec{t}|^n s))$.

This claim establishes that given any $\vec{t} \in [0,1]^n$ and $i \in N$ there exists some $\vec{\lambda}$ such that $\{a_i(\vec{t}|^i s) | s \in [0,1]\} \subseteq \{S_i(\vec{\lambda}|^i \lambda) | \lambda \in \mathcal{L}\}$ (actually, equality holds, but we need only establish inclusion). This fact, combined with the fact that $\vec{S}$ is strictly strategy-proof, implies that $\lambda_i \cdot a_i(\vec{z}|^i s)$ has a unique maximizer for each $\lambda_i \in \mathcal{L}$. If the maximizer of $\lambda_i \cdot a_i(\vec{z}|^i s)$ is $\bar{s}$, then $\bar{s}$ remains the unique maximizer of $\lambda_i \cdot a_i(\vec{t}|^i s)$ for all $\vec{t}$ such that for all $j \neq i$ $MIN[t_j, z_j] < \bar{s} \Rightarrow t_j = z_j$ (This follows from the continuity in ϵ of $\vec{a}(\vec{z} + \epsilon(\vec{t} - \vec{z})|^i s)$ and the fact that $a_i(\vec{z} + \epsilon(\vec{t} - \vec{z})|^i \bar{s}) = a_i(\vec{z}|^i \bar{s})$, so if $\bar{s}$ is not the maximizer of $\lambda_i \cdot a_i(\vec{t}|^i s)$ then there exists some $\bar{\epsilon}$ such that $\lambda_i \cdot a_i(\vec{z} + \epsilon(\vec{t} - \vec{z})|^i s)$ does not have a unique maximizer.). Furthermore, by construction, $a_i(\vec{s})$ is independent of s_j as long as $s_j > s_i$ and $\sum_{i=1}^n a_i(\vec{z}) \in \mathcal{P}$. Therefore, the function $\vec{a}$ obeys all of the conditions for a generalized serial function. □

6.3 Proof of Corollaries

6.3.1 Proof of Corollary 2

Proof: If there were such an $\vec{S}$, then it would be associated with some family in $\mathcal{F}$. However, $\mathcal{F}$ is empty for IRS $\mathcal{P}$'s because one cannot find a curve $b : [0,1] \mapsto \mathcal{P}$ that has a unique maximizer of $\lambda \cdot b(s)$ for each $\lambda \in \mathcal{L}$. More specifically, consider any profile $\vec{\lambda} \in \mathcal{Z}(\vec{S})$, labelled so that n is the tail element, with $\lambda_n^\alpha = \frac{\partial A^\nu}{\partial x^\alpha}$ for some $\nu \in P$. Choose any $v \in TS_n(\vec{\lambda})$ and define $g(\epsilon) = S_n(\vec{\lambda}|^n \lambda_n + \epsilon v)$. Taking the second derivative of the constraint equation $A^\nu(\sum_{i=1}^n S_i(\vec{\lambda}|^n \lambda_i + \epsilon v)) = 0$ yields $\sum_{\alpha,\beta=1}^m \frac{\partial A^\nu}{\partial x^\alpha} \frac{\partial A^\nu}{\partial x^\beta} \frac{\partial g^\alpha}{\partial x} \frac{\partial g^\beta}{\partial x} + \lambda \cdot g''(0) = 0$. The second term is strictly negative and the first, by the IRS condition, is nonpositive. Contradiction. □

6.3.2 Proof of Corollary 3

Proof: We know that any anonymous, nonconstant, and recursively Z-smooth $\vec{S} \in \tilde{\mathcal{S}}$ must be a generalized serial mechanism. Assume that such an $\vec{S}$ is differentiable (C^2) everywhere. Consider some profile $\vec{\lambda} \in \mathcal{Y}(\vec{S}^{1;2;\ldots;n-2;T(n-1)})$ where clique $T(n-1)$ is the tail element and $\lambda_n = \lambda_{n-1} \in NA(\vec{\lambda})$. Consider any $v \in TS_{n-1}(\vec{\lambda})$ which points toward the tail. Define $g(\epsilon) = S_n(\vec{\lambda}|^n \lambda_n + \epsilon v)$. Recall that the smoothness property implies that $g'(0) \cdot v > 0$. Taking the derivative of the relation $g'(\epsilon) \cdot (\lambda + \epsilon v) = 0$ then yields $g''(0) \cdot \lambda_n < 0$. Since $S_n(\vec{\lambda}|^{n-1} \lambda_{n-1} + \epsilon v) = S_n(\vec{\lambda})$ for $\epsilon > 0$, the continuity of the first and second derivatives requires that $\frac{\partial S_n}{\partial \lambda_{n-1}^\alpha}(\vec{\lambda}) = 0$, $\frac{\partial^2 S_n}{(\partial \lambda_{n-1}^\alpha)^2}(\vec{\lambda}) = 0$, and $\frac{\partial^2 S_n}{\partial \lambda_{n-1}^\alpha \partial \lambda_n^\beta}(\vec{\lambda}) = 0$. Taking second derivatives of the expression $A^\nu(\sum_{k=1}^n S_k(\vec{\lambda}|^{n-1;n} \lambda_{n-1} + \epsilon v, \lambda_{n-1} +$

$\delta v)) = 0$ evaluated at $\epsilon = \delta = 0$, we find that $\sum_{\alpha}^{m} \frac{\partial A^\nu}{\partial x^\alpha} \frac{\partial^2 g^\alpha}{(\partial\epsilon)^2} = 0$. Since this must hold for all $\nu \in P$, $g''(0) \in TA(\vec{\lambda})$. But $\lambda_{n-1} \in NA(\vec{\lambda})$ which would imply that $\lambda_{n-1} \cdot g''(0) = 0$. Contradiction. $\square$

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8 References

S. Barberà and M. Jackson (1991): "A Characterization of Strategy-Proof Social Choice Functions for Economies with Pure Public Goods," mimeo.

S. Barberà and M. Jackson (1992): "Strategy-Proof Exchange," mimeo.

S. Barberà and B. Peleg (1990): "Strategy-Proof Voting Schemes with Continuous Preferences," *Social Choice and Welfare*, 7, 31-38.

S. Barberà, H. Sonnenschein, and L. Zhou (1991): "Voting by Committees," *Econometrica*, 59, 595-609.

E. Maskin (1985): "Theory of Implementation in Nash Equilibrium," in *Social Goals and Social Organization*, ed. by L. Hurwicz, D. Schmeidler, and H. Sonnenschein. Cambridge: Cambridge University Press, 173-204.

H. Moulin (1980): "On Strategy-Proofness and Single Peakedness," *Public Choice*, 35, 437-455.

H. Moulin (1991): "Excludable Public Goods and the Free Rider Problem," mimeo.

H. Moulin and S. Shenker (1992): "Serial Cost Sharing," *Econometrica*, 60, 1009-1037.

M. Satterthwaite and H. Sonnenschein (1981): "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Reviews of Economic Studies*, 48, 587-587.

S. Shenker (1992): "Some Technical Results on Continuity, Strategy-proofness, and Related Strategic Concepts," mimeo.

Y. Sprumont (1992): "Continuous Strategyproof Mechanism for Sharing Private Goods," CRDE #1192.

L. Zhou (1991a): "Impossibility of Strategy-Proof Mechanisms in Economies with Pure Public Goods," *Reviews of Economic Studies*, 58, 107-119.

L. Zhou (1991b): "Inefficiency of Strategy-Proof Allocation Mechanisms in Pure Exchange Economies," *Social Choice and Welfare*, 8, 1-8.