Some Technical Results on Continuity, Strategy-proofness, and Related Strategic Concepts

Scott Shenker[1]
Palo Alto Research Center
Xerox Corporation
3333 Coyote Hill Drive
Palo Alto, CA 94304
(shenker@parc.xerox.com)

Abstract

There are many concepts of nonmanipulability, including strategy-proofness (and its several coalitional variants), Nash implementability, self Nash implementability, and nonbossiness. We explore the relationships between these concepts in the context of continuous social choice functions (SCF's). For any continuous SCF on a continuously and monotonically closed domain, we find that strategy-proofness implies strict strategy-proofness, and that strategy-proofness and nonbossiness imply Maskin monotonicity. On the domain of monotonic, nonlocally satiated, and strictly convex preferences over $\Re^m_+$, any strictly strategy-proof SCF is continuous, and any continuous, strategy-proof, and nonbossy SCF is strictly coalitionally strategy-proof. On the domain of monotonic, nonlocally satiated, and convex preferences over $\Re^m_+$, any coalitionally strategy-proof SCF is Maskin monotonic, and any strategy-proof and nonbossy social choice function is completely characterized by its values on the set of linear preferences.

[1] I gratefully acknowledge many helpful discussions with Hervé Moulin.

1 Introduction

This paper presents a few technical results concerning the impact of continuity on the relationships between various forms of strategy-proofness for social choice functions. The two unusual foci of this paper are that we stress strategic concepts that are more demanding than strategy-proofness, and we investigate the implications of continuity. In Section 2, we present various definitions of nonmanipulability of social choice functions. In general, there are few nontrivial relationships between these concepts. When the domain is restricted to monotonically closed domains, as we discuss in Section 3, a few nontrivial relationships emerge. In addition, a result relating fairness and nonmanipulability is presented in Section 4. Section 5 contains a general discussion of the impact of requiring a social choice function to be continuous when the domain has a property we call *continuous closure*. Finally, in Section 6, we discuss the specific domains of monotonic, nonlocally satiated, and either convex or strictly convex preferences over $\Re^m_+$.

We consider a resource allocation problem with n agents, with some set of possible allocations $\mathcal{A}$, and a domain $\mathcal{U}$ of utility functions defined over this set. While we will use the functional representation of utilities rather than the more cumbersome binary relation representation, all of our definitions and our results are purely ordinal. A social choice function is a function $S : \mathcal{U}^n \mapsto \mathcal{A}$. Our analysis applies to both the private goods and the public goods cases. In the private goods case, there is some set $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}^n$ and the utilities in $\mathcal{U}$ are defined over this set $\mathcal{B}$. Without loss of generality, we will use the private goods notation, and let $S_i(\vec{U})$ denote the goods given to agent i; in the public goods context we always have $S_i(\vec{U}) = S_j(\vec{U})$ for all i, j. Also, we will use the notation that for any vector $\vec{x}$, $(\vec{x}|^i z)$ denotes the vector with $(\vec{x}|^i z)_j = x_j$ for all $j \neq i$ and $(\vec{x}|^i z)_i = z$.

2 Definitions of Nonmanipulability

When working in a noncooperative context, it is natural to require that the social choice functions we consider be compatible with user incentives, and thus be invulnerable to strategic manipulation. There are many different definitions of nonmanipulability, and we review some of them here.

2.1 Strategy-Proofness

The first, and most common, nonmanipulability criterion is to require that S be strategy-proof. Strategy-proofness requires that the truth be a dominant strategy. Equivalently, strategy-proofness requires that the truth is always a Nash equilibrium of S: S is *strategy-proof* (SP) if, for any $\vec{U} \in \mathcal{U}^n$, $U_i(S_i(\vec{U})) \geq U_i(S_i(\vec{U}|^i V_i))$ for all $V_i \in \mathcal{U}$. Strategy-proofness is often treated in the literature as the epitome of nonmanipulability. However, there are several ways in which SP social choice functions are vulnerable to manipulation.

First, strategy-proofness does not require that S provide incentives to an agent to announce the truth; it merely requires that a single deviating agent be no better than indifferent between the resulting outcome and the truthful one. Thus, while the truth must be a Nash equilibrium, it need not be a strict[2] Nash equilibrium. To rectify this, one can define a slightly stronger concept, namely *Strictly Strategy-Proof* (SSP), where the truth is always a strict Nash equilibrium: S is SSP if, for any $\vec{U} \in \mathcal{U}^n$ and $V_i \in \mathcal{U}$, $S_i(\vec{U}) \neq S_i(\vec{U}|^i V_i) \Rightarrow U_i(S_i(\vec{U})) > U_i(S_i(\vec{U}|^i V_i))$. With SSP social choice functions, a single deviating agent cannot achieve any nontruthful outcome equivalent to the truthful outcome. Consider some profile $\vec{U}$ and let V_i be such that for all allocations x, $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow V_i(x) \leq V_i(S_i(\vec{U}))$. If S is SSP, then $S_i(\vec{U}|^i V_i) = S_i(\vec{U})$; however, the conclusion does not follow if S is merely SP. This property is closely related to Maskin monotonicity (in fact, in the public goods case it implies it), to which we return later in Section 2.2.

Another form of manipulation not dealt with by strategy-proofness or strict strategy-proofness is manipulation by coalitions. One can modify the traditional strategy-proof concept to account for such coalitional behavior; below we present three definitions of strategy-proofness with respect to coalitions.

Definition 1 *Consider any pair $\vec{U}, \vec{V} \in \mathcal{U}^n$ and define $E = \{i | U_i \neq V_i\}$. S is* weakly coalitionally strategy-proof *(WCSP) if, when E is nonempty, there always exists some $j \in E$ such that $U_j(S_j(\vec{V})) \leq U_j(S_j(\vec{U}))$. S is* coalitionally strategy-proof *(CSP) if either $U_j(S_j(\vec{U})) = U_j(S_j(\vec{V}))$ for all $j \in E$, or there exists some $j \in E$ such that $U_j(S_j(\vec{V})) < U_j(S_j(\vec{U}))$. S is* strictly coalitionally strategy-proof *(SCSP) if $S(\vec{U}) \neq S(\vec{V}) \Rightarrow$there exists $j \in E$ such that $U_j(S(\vec{V})) < U_j(S(\vec{U}))$.*

WCSP[3] merely requires that not all members of the deviating coalition can strictly gain by deviating. CSP requires that a coalition deviating from the truth be no better than indifferent between the resulting outcome and the truthful outcome; i.e., the truth is a strong equilibrium of S. SCSP requires that there is no other outcome that is equivalent or better, in the eyes of the deviating coalition, to the truthful outcome. Thus, for an SCSP S, the truth is a strict strong equilibrium.

The third manipulability problem with the strategy-proof paradigm is that while strategy-proofness requires that the truth have certain strategic properties, it does not preclude the existence of other Nash equilibria of the social choice function S. For instance, consider a public goods case with two outcomes a, b, and consider the S that, at a given profile $\vec{U}$, chooses outcome a unless all agents strictly prefer outcome b, in which case it chooses outcome b. The truth is always a strong equilibrium for this S. However, whenever $\vec{V}$ has two agents who strictly prefer outcome a, $\vec{V}$ is a Nash equilibrium of S (no matter what the true profile is). The existence of these *spurious* Nash equilibria is addressed by the Nash implementation paradigm, to which we now turn.

2.2 Nash Implementability

An alternative to the strategy-proofness criterion for nonmanipulability is to require that there be some mechanism which Nash implements S. We say that S is *Nash implementable* if there is

[2]Here we use the term *strict* to mean that all outcomes reachable by unilateral deviation are either the equilibrium outcome, or are strictly inferior for the deviating agent. This property is termed *quasi-strict* in Fudenberg-Tirole (1991).

[3]This is also referred to as Group Strategy-Proof; see Muller-Satterthwaite (1985).

some set of message spaces $\mathcal{M}_i$ and a mechanism $G : \mathcal{M}_1 \times \mathcal{M}_2 \times \ldots \times \mathcal{M}_n \mapsto \mathcal{A}$ such that, for each profile $\vec{U}$, the set of Nash equilibria $NE(\vec{U})$ is nonempty and each element $\vec{\sigma} \in NE(\vec{U})$ obeys $G(\vec{\sigma}) = S(\vec{U})$. The concept of Nash implementability is closely related to that of Maskin monotonicity (MM), since all Nash implementable S are Maskin monotonic.

Definition 2 *We say that S is* Maskin monotonic *(MM) if $S(\vec{V}) = S(\vec{U})$ whenever $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow V_i(x) \leq V_i(S_i(\vec{U}))$ for all allocations x and all i.*

The following result is from Maskin (1985).

Lemma 1 [Maskin] *If S is Nash implementable, then S is Maskin monotonic.*

As opposed to strategy-proofness, the paradigm of Nash implementability does preclude the existence of other Nash outcomes. However, a strategy-proof mechanism is truly decentralizable, in that each agent can choose to tell the truth without coordinating with other agents. In contrast, the Nash implementation paradigm requires coordination among the agents to reach the appropriate equilibrium; common-knowledge and Cournot tattonement are two of the commonly cited coordination devices. We now turn to combining the strength of the strategy-proof paradigm (namely that truth is a dominant strategy, so a naive agent can always announce the truth without fear of playing suboptimally) with the strength of the Nash implementable paradigm (namely that there is no other Nash outcome, so if no agent plays individually suboptimally the outcome must be the truthful one).

2.3 Self Nash Implementing

We say that S is self Nash implementing (SNI) if S Nash implements itself. Formally,

Definition 3 *S is* self Nash implementing *(SNI) if, for all $\vec{U}$, there is at least one Nash equilibrium $\vec{W}$ of S and $S(\vec{U}) = S(\vec{W})$ for each such Nash equilibrium $\vec{W}$.*

Note that we have not, *a priori*, included the requirement that the truth be one of the Nash equilibria; that will follow given certain conditions on the domain $\mathcal{U}$ as we shall see in the Section 3. However, in the notion of SNI we have not provided for any coalition-proof properties. One could define variants of SNI, like self strong implementing, which use more stringent equilibrium concepts. While such self strong implementing social choice functions would be invulnerable to deviations by coalitions, the concept would not preclude the existence of non-strong Nash equilibrium outcomes that were not truthful.

An example of a social choice function that is SNI is one for two agents with strictly monotonic linear preferences over $\Re^2$. With the notation $U_i(x, y) = \lambda_i x + y$, S is given by: $S_1(U_1, U_2) = (\lambda_1, -\frac{{\lambda_1}^2}{2} - \lambda_2)$ and $S_2(U_1, U_2) = (\lambda_2, -\frac{{\lambda_2}^2}{2} - \lambda_1)$. Here, the truth is always the unique Nash equilibrium, and S, in addition to being SNI, is also SSP (and also nonbossy, which we will define in Section 2.4). However, S is vulnerable to coalitional manipulation, as it is not even WCSP. This can be seen most easily in the case where $U_1 = U_2$ and the utilities are maximized when both agents report utilities with λ's one less than the truth: $\tilde{\lambda}_i = \lambda_i - 1$.

2.4 Nonbossiness

In the private goods case, there is another form of manipulation called *bossiness* which is not directly addressed by the concepts above (though it is implied by some of them); an agent can boss another agent around if, by changing her announced utility, she can change another agent's allocation without changing her own allocation. This problem, of course, does not arise in the public goods case. Satterthwaite-Sonnenschein (1981) introduce the notion of *nonbossiness* (NB): S is nonbossy if $S_i(\vec{U}) = S_i(\vec{U}|^i V_i) \Rightarrow S(\vec{U}) = S(\vec{U}|^i V_i)$. We can generalize this concept to coalitions: S is *coalitionally nonbossy* (CNB) if, for any pair $\vec{U}, \vec{V} \in \mathcal{U}^n$, $S(\vec{U}) = S(\vec{V})$ whenever $S_i(\vec{U}) = S_i(\vec{V})$ for each i such that $U_i \neq V_i$. Note that any SCSP S satisfies a stronger condition than this: if $U_i(S_i(\vec{U})) = U_i(S_i(\vec{V}))$ for each i such that $U_i \neq V_i$, then $S(\vec{U}) = S(\vec{V})$. Of course, any public goods social choice function, whether strategy-proof or not, is CNB.

The combination of strict strategy-proofness and nonbossiness yields a particularly appealing strategic property: S is SSP and NB if and only if $S(\vec{U}|^i V_i) \neq S(\vec{U}) \Rightarrow U_i(S_i(\vec{U}|^i V_i)) < U_i(S_i(\vec{U}|^i V_i))$. Thus, if S is SSP and NB, then if any agent's unilateral deviation changes the allocation vector, then that agent must be strictly worse off. This combination of SSP and NB, which can be seen as the noncoalitional version of SCSP, will play a central role in some of our later theorems.

In the absence of any restrictions on the domains or on the social choice functions, there are few nontrivial relationships between the nonmanipulability concepts introduced so far. In the next two sections we investigate the implications of monotonic closure of the domain, and of continuity of the social choice function.

3 Monotonic Closure

We now consider the property of *monotonic closure* (MC), a concept originally called *rich domain* in Dasgupta-Hammond-Maskin (1979) and then rechristened in Maskin (1985).

Definition 4 [Maskin] *A domain of utility functions $\mathcal{U}$ is monotonically closed (MC) if, for all pairs $U, V \in \mathcal{U}$, and all pairs $x, y \in \mathcal{B}$ such that (1) $U(x) \geq U(y) \Rightarrow V(x) \geq V(y)$, and (2) $U(x) > U(y) \Rightarrow V(x) > V(y)$, there exists some $W \in \mathcal{U}$ such that for all allocations $z \in \mathcal{B}$, (3) $U(x) \geq U(z) \Rightarrow W(x) \geq W(z)$, and (4) $V(y) \geq V(z) \Rightarrow W(y) \geq W(z)$.*

The following result and its proof are almost identical to that of Maskin (see Theorem 7 in Maskin (1985)). While it has been well-established that Nash implementability in monotonically closed domains implies SP, it turns out that Nash implementability implies a slightly stronger property.

Theorem 1 *If $\mathcal{U}$ is MC and S is Maskin monotonic, then S is weakly coalitionally strategy-proof and coalitionally nonbossy.*

Proof: First, consider two profiles $\vec{U}$ and $\vec{V}$. Let $E = \{i | U_i \neq V_i\}$. Assume that $S_i(\vec{V}) = S_i(\vec{U})$ for each $i \in E$ but $S(\vec{V}) \neq S(\vec{U})$. Define $\vec{W}$ by $W_i = U_i = V_i$ for each $i \notin E$ and let $W_i(x)$ be such that $U_i(z) \leq U_i(S_i(\vec{U})) \Rightarrow W_i(z) \leq W_i(S_i(\vec{U}))$ and $V_i(z) \leq V_i(S_i(\vec{U})) \Rightarrow W_i(z) \leq W_i(S_i(\vec{U}))$ for each $i \in E$ (the existence of such W_i is guaranteed by the MC condition). Applying Maskin monotonicity, we must then have $S(\vec{V}) = S(\vec{W}) = S(\vec{U})$, which contradicts our assumption. Therefore, S is coalitionally nonbossy.

To see that S is weakly coalitionally strategy-proof, consider again the two profiles $\vec{U}$ and $\vec{V}$ with $E = \{i | U_i \neq V_i\}$. Assume that, contrary to the WCSP condition, $U_i(S_i(\vec{V})) > U_i(S_i(\vec{U}))$ for each $i \in E$. The monotonic closure of $\mathcal{U}$ requires that, for each $i \in E$, there exists some W_i such that for all allocations x, (1) $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow W_i(x) \leq W_i(S_i(\vec{U}))$ and (2) $V_i(x) \leq V_i(S_i(\vec{V})) \Rightarrow W_i(x) \leq W_i(S_i(\vec{V}))$. If we define the rest of the vector $\vec{W}$ by $W_i \equiv U_i = V_i$ for each $i \notin E$, then, by applying Maskin monotonicity, we find that $S(\vec{W}) = S(\vec{U})$ and $S(\vec{W}) = S(\vec{V})$ which contradicts that assumption that $U_i(S_i(\vec{V})) > U_i(S_i(\vec{U}))$ for each $i \in E$. □

Corollary 1 *If $\mathcal{U}$ is MC and S is SNI, then S is CNB and WCSP.*

Proof: The conclusion follows immediately from the observation that SNI $\Rightarrow$ NI and NI $\Rightarrow$ MM. □

In particular, even though it was not part of the definition of self Nash implementing, in monotonically closed domains if S is SNI then the truth is always a Nash equilibrium. In addition to the result that MM $\Rightarrow$ CNB+WCSP, we have the partial converse that SSP+NB $\Rightarrow$ MM.

Corollary 2 *If $\mathcal{U}$ is MC and S is SSP and NB, then S is MM.*

Proof: Consider some $\vec{U}$ and some $\vec{V}$ with $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow V_i(x) \leq V_i(S_i(\vec{U}))$ for all allocations x and all i. From SSP, we know that either $S_i(\vec{U}|^i V_i) = S_i(\vec{U})$, or $U_i(S_i(\vec{U}|^i V_i)) < U_i(S_i(\vec{U}))$ and $V_i(S_i(\vec{U}|^i V_i)) > V_i(S_i(\vec{U}))$. The second case is impossible, since by assumption $U(x) < U(S_i(\vec{U})) \Rightarrow V(x) \leq V(S_i(\vec{U}))$. Thus, the first case must hold; nonbossiness then implies that $S(\vec{U}|^i V_i) = S(\vec{U})$. Repeating this process in turn for each i yields the conclusion that $S(\vec{U}) = S(\vec{V})$. □

Theorem 2 *If $\mathcal{U}$ is MC, S is SSP and NB, and $n = 2$, then S is CSP.*

Proof: From Corollary 2, we know that S must be MM, and from Theorem 1 we know that S must be WCSP. Assume to the contrary that S is not CSP; i.e., assume that there is some (U_1, U_2) and some (V_1, V_2) such that $U_1(S_1(V_1, V_2)) > U_1(S_1(U_1, U_2))$ and $U_2(S_2(V_1, V_2)) = U_2(S_2(U_1, U_2))$. From the monotonically closed condition, we can find some W_1 such that $S(W_1, U_2) = S(U_1, U_2)$ and $S(W_1, V_2) = S(V_1, V_2)$. Then, we have that $U_1(S_1(W_1, V_2)) > U_1(S_1(W_1, U_2))$ and $U_2(S_2(W_1, V_2)) = U_2(S_2(W_1, U_2))$. Since S is SSP, we must have $S_2(W_1, V_2) = S_2(W_1, U_2)$. The NB condition yields $S_1(W_1, V_2) = S_1(W_1, U_2)$, which is in contradiction to the fact that $U_1(S_1(W_1, V_2)) > U_1(S_1(W_1, U_2))$. □

The question of whether this result holds for larger n is open. Some domains allows a stronger property than MC, which we will call *strict monotonic closure*.

Definition 5 *A domain of utility functions $\mathcal{U}$ is strictly monotonically closed (SMC) if it is MC and if, for all pairs $U, V \in \mathcal{U}$, and all pairs $x, y \in \mathcal{B}$ such that (1) $U(x) \geq U(y) \Rightarrow V(x) \geq V(y)$, and (2) $U(x) > U(y) \Rightarrow V(x) > V(y)$, there exists some $W \in \mathcal{U}$ such that for all allocations $z \in \mathcal{B}$, (3) ($z \neq x$ and $U(x) \geq U(z)$) $\Rightarrow W(x) > W(z)$, and (4) ($z \neq y$ and $V(y) \geq V(z)$) $\Rightarrow W(y) > W(z)$.*

In this case, we have the slightly stronger result that SP+NB $\Rightarrow$ WCSP+CNB.

Theorem 3 *If $\mathcal{U}$ is SMC and S is SP and NB then S is WCSP and CNB.*

Proof: First, consider two profiles $\vec{U}$ and $\vec{V}$. Let $E = \{i | U_i \neq V_i\}$. Assume that $S_i(\vec{V}) = S_i(\vec{U})$ for each $i \in E$ but $S(\vec{V}) \neq S(\vec{U})$. Define $\vec{W}$ by $W_i = U_i = V_i$ for each $i \notin E$ and let $W_i(x)$ be such that, for all $z \neq S_i(\vec{U})$, $U(z) \leq U(S_i(\vec{U})) \Rightarrow W(z) < W(S_i(\vec{U}))$ and $V(z) \leq V(S_i(\vec{U})) \Rightarrow W(z) < W(S_i(\vec{U}))$ for each $i \in E$ (the existence of such W_i is guaranteed by the SMC condition). For any $i \in E$, consider $S(\vec{U}|^i W_i)$. From SP we must have $U_i(S_i(\vec{U}|^i W_i)) \leq U_i(S_i(\vec{U}))$ and $W_i(S_i(\vec{U}|^i W_i)) \geq W_i(S_i(\vec{U}))$. From the construction of W_i we must then have $S_i(\vec{U}|^i W_i) = S_i(\vec{U})$, and by NB we have $S(\vec{U}|^i W_i) = S(\vec{U})$. Starting with the profile $(\vec{U}|^i W_i)$ and sequentially applying this reasoning to each element in E, we find that $S(\vec{W}) = S(\vec{U})$. Similarly, $S(\vec{W}) = S(\vec{V})$, which contradicts our assumption. Therefore, S is coalitionally nonbossy.

To see that S is weakly coalitionally strategy-proof, consider again the two profiles $\vec{U}$ and $\vec{V}$ with $E = \{i | U_i \neq V_i\}$. Assume that, contrary to the WCSP condition, $U_i(S_i(\vec{V})) > U_i(S_i(\vec{U}))$ for each $i \in E$. The strict monotonic closure property of $\mathcal{U}$ requires that, for each $i \in E$, there exists some W_i such that (1) for all allocations $x \neq S_i(\vec{U})$, $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow W_i(x) < W_i(S_i(\vec{U}))$, and (2) for all allocations $x \neq S_i(\vec{V})$, $V_i(x) \leq V_i(S_i(\vec{V})) \Rightarrow W_i(x) < W_i(S_i(\vec{V}))$. Define the rest of the vector $\vec{W}$ by $W_i \equiv U_i = V_i$ for each $i \notin E$. Then, by using reasoning similar to that in the previous paragraph, we find that $S(\vec{W}) = S(\vec{U})$ and $S(\vec{W}) = S(\vec{V})$ which contradicts that assumption that $U_i(S_i(\vec{V})) > U_i(S_i(\vec{U}))$ for each $i \in E$. $\square$

The fact that strategy-proofness and nonbossiness implies weakly coalitional strategy-proofness was observed by Barberà and Jackson (1992, Lemma 3) for exchange economies.

4 Fairness

We now temporarily turn from nonmanipulability to fairness. An *a priori* notion of fairness is that S be anonymous (i.e., fully symmetric in its arguments). An *a posteriori* definition of fairness is that that the resulting alloctions be envy-free. It turns out that for a Maskin monotonic S, an even weaker form of *a priori* fairness implies *a posteriori* fairness. We say that S is weakly anonymous if $S_i(\vec{U}) = S_j(\vec{U})$ whenever $U_i = U_j$. It turns out that any weakly anonymous S which is Maskin monotonic produces envy-free allocations. This result is very similar to that of Moulin (1991), where it is shown that CSP and weakly anonymous social choice functions produce envy-free allocations[4] Here, because we have the slightly different strategic conditions, we must rederive the result.

[4]In fact, an even weaker notion of anonymity is used: $U_i = U_j \Rightarrow U_i(S_i(\vec{U})) = U_i(S_j(\vec{U}))$.

Theorem 4 *If $\mathcal{U}$ is MC and S is weakly anonymous and Maskin monotonic, then for all i, j and all $\vec{U} \in \mathcal{U}^n$ with $S_i(\vec{U}) \neq S_j(\vec{U})$, (1) $U_i(S_i(\vec{U})) \geq U_i(S_j(\vec{U}))$ and (2) $U_i(S_i(\vec{U})) = U_i(S_j(\vec{U})) \Rightarrow U_j(S_i(\vec{U})) < U_j(S_j(\vec{U}))$.*

Proof: Assume that we have some $\vec{U}$ and some i, j such that $S_i(\vec{U}) \neq S_j(\vec{U})$ and either $U_i(S_j(\vec{U})) > U_i(S_i(\vec{U}))$, or $U_i(S_j(\vec{U})) = U_i(S_i(\vec{U}))$ and $U_j(S_i(\vec{U})) = U_j(S_j(\vec{U}))$. By the monotonic closure of $\mathcal{U}$, we can find some $V \in \mathcal{U}$ such that $U_i(x) \leq U_i(S_i(\vec{U})) \Rightarrow V(x) \leq V(S_i(\vec{U}))$ and $U_j(x) \leq U_j(S_j(\vec{U})) \Rightarrow V(x) \leq V(S_j(\vec{U}))$. Then, Maskin monotonicity requires that $S(\vec{U}|^{\{i,j\}}V) = S(\vec{U})$, and then the weak anonymity requires that $S_i(\vec{U}) = S_j(\vec{U})$, violating the assumption that $S_i(\vec{U}) \neq S_j(\vec{U})$. □

5 Continuity

We now proceed to discuss the role of continuity and how it affects the strategic properties of social choice functions. Given some topology on $\mathcal{B}$, we induce a topology on $\mathcal{U}$ via the condition that a sequence of $V^t \in \mathcal{U}$ converges to U if and only if the set of upper contours of the V^t converges to the set of upper contours of U. We now introduce the notion of *continuously closed* (CC) domains below.

Definition 6 *We say that a domain of utility functions $\mathcal{U}$ and a set of allocations $\mathcal{B}$ equipped with some topology is* continuously closed *(CC) if for all distinct pairs of allocations x, y there is some $U \in \mathcal{U}$ such that $U(x) \neq U(y)$ and, for each pair of utility functions $U, V \in \mathcal{U}$ and pair of allocations x, y such that $U(x) = U(y)$ and $V(x) > V(y)$, there exists a sequence of utility functions U^t with $\lim_{t\to\infty} U^t = U$ and an open neighborhood $\mathcal{N}$ around y such that, for all $z \in \mathcal{N}$, $U^t(z) \geq U^t(x) \Rightarrow U(z) > U(x)$.*

In what follows, we assume that $\mathcal{U}$ is continuously closed. The addition of continuity to the strategic properties yields the following results. A similar result was derived, for exchange economies, by Sprumont (1992).

Theorem 5 *If $\mathcal{U}$ is MC and CC and S is continuous and SP, then S is SSP.*

Proof: Consider a continuous and SP S and some $\vec{U} \in \mathcal{U}^n$. Assume, to the contrary, that there is some i and some $W_i \in \mathcal{U}$ such that $U_i(S_i(\vec{U})) = U_i(S_i(\vec{U}|^iW_i))$ but $S_i(\vec{U}) \neq S_i(\vec{U}|^iW_i)$. Note that $W_i(S_i(\vec{U})) \leq W_i(S_i(\vec{U}|^iW_i))$. If the inequality is strict, then there clearly exists some V such that $V(S_i(\vec{U}|^iW_i)) > V(S_i(\vec{U}))$, since $V = W_i$ is an example. If equality holds, it is also true that, by switching if need be the roles of U_i and W_i, we can assume without loss of generality that there exists some V such that $V(S_i(\vec{U}|^iW_i)) > V(S_i(\vec{U}))$. Thus, there exists some sequence U^t, $\lim_{t\to\infty} U^t = U_i$, and some neighborhood $\mathcal{N}$ around $S_i(\vec{U})$ such that, for all $z \in \mathcal{N}$, $U^t(z) \geq U^t(S_i(\vec{U}|^iW_i)) \Rightarrow U_i(z) > U_i(S_i(\vec{U}))$. Note that $U_i(S_i(\vec{U}|^iW_i)) = U_i(S_i(\vec{U})) \geq U_i(S_i(\vec{U}|^iU^t))$ and, for sufficiently large t, $S_i(\vec{U}|^iU^t) \in \mathcal{N}$; since $U_i(S_i(\vec{U}|^iU^t)) \leq U_i(S_i(\vec{U}|^iW_i))$ we must have by construction $U^t(S_i(\vec{U}|^iU^t)) < U^t(S_i(\vec{U}|^iW_i))$ for all sufficiently large t. This obviously contradicts SP. □

As a corollary to this, we have the following result.

Corollary 3 *Assume that $\mathcal{U}$ is MC and CC. For a continuous S, the following three conditions are equivalent:*

1. *S is SP and NB*
2. *S is SSP, WCSP, and CNB*
3. *S is Maskin monotonic*

Proof: $1 \Rightarrow 3$: continuity plus SP implies SSP, and SSP+NB implies MM. $3 \Rightarrow 2$: we know, from Theorem 1, that MM implies WCSP+CNB, and then continuity plus SP implies SSP. $2 \Rightarrow 1$: this follows trivially from the definitions. $\square$

6 Results on Two Specific Domains

6.1 Some Preliminary Definitions and Results

Let $\mathcal{U}^c$ be the set of continuous, nondecreasing in each dimension, nonlocally satiated, and quasi-concave functions $U : \Re^m_+ \mapsto \Re$. Let $\mathcal{U}^{sc}$ be the set of those elements of $\mathcal{U}^c$ that are strictly quasi-concave. We use the standard topology on $\Re^m_+$, and the induced standard topologies on $\mathcal{U}^c$ and $\mathcal{U}^{sc}$. Note that, in this topology, if $\lim_{t\to\infty} U^t = U$ in the induced topology then we can find another sequence $\tilde{U}^t$ of utility functions representing the same preferences such that $\lim_{t\to\infty} \tilde{U}^t(x) = U(x)$ for all allocations x.

The domains $\mathcal{U}^c$ and $\mathcal{U}^{sc}$ are monotonically closed, and in fact $\mathcal{U}^{sc}$ is strictly monotonically closed. This can be seen by Example 3.1.2 in Dasgupta-Hammond-Maskin (1979), where a similar result is established for continuous, strictly convex, strictly monotonic preferences. Note that our domains $\mathcal{U}^c$ and $\mathcal{U}^{sc}$ are also continuously closed.

Lemma 2 *$\mathcal{U}^c$ and $\mathcal{U}^{sc}$ are continuously closed.*

Proof: The following proof applies for both $\mathcal{U} = \mathcal{U}^c$ and $\mathcal{U} = \mathcal{U}^{sc}$. Clearly, for all distinct pairs of allocations x, y, there is some utility $U \in \mathcal{U}$ such that $U(x) \neq U(y)$. Consider some pair of utility functions $U, V \in \mathcal{U}$ and a pair of allocations x, y such that $U(x) = U(y)$ but $V(x) > V(y)$. Then, $(y - x) \notin \Re_+$. Let $W(z) = z \cdot (x - y)_+$ where the notation $(x - y)^\alpha_+ = MAX[0, (x - y)^\alpha]$; then, $W(x) > W(y)$. Define $u = W - U$ and let $\mathcal{N}$ be any sufficiently small open neighborhood of y; $U + \delta u \in \mathcal{U}$ for all $0 \leq \delta < 1$, $\lim_{\delta\to 0^+} U + \delta u = U$, and (3) $u(x) > u(z)$ for all $z \in \mathcal{N}$. Thus, if we set $U^t = U + \frac{1}{2t} u$, we have a sequence of utility functions which, for $t = 1, 2, \ldots$, satisfies the conditions used in the definition of continuous closure. $\square$

Before proceeding, we need a few basic definitions. Let $\mathcal{S}^c$ denote the set of functions $S : (\mathcal{U}^c)^n \mapsto \mathcal{A}$, and let $\mathcal{S}^{sc}$ denote the set of functions $S : (\mathcal{U}^{sc})^n \mapsto \mathcal{A}$. Let $\mathcal{L}$ denote the linear subset of $\mathcal{U}^c$, which is isomorphic to $\Re_+^m - \{0^m\}$, and let $L_\lambda(x) = \lambda \cdot x$. Furthermore, let $\mathcal{V}$ denote the subset of $\mathcal{U}^{sc}$ which is twice continuously differentiable (C^2) and has $\sum_{\alpha=1}^m \|\frac{\partial U}{\partial x^\alpha}\| > 0$ everywhere in $\Re_+^m$. Note that $\mathcal{V}$ is dense in $\mathcal{U}^c$ and in $\mathcal{U}^{sc}$. Let $f^a(\epsilon) \equiv -a^{-1}\epsilon + a^{-3}(1 - e^{-a^2\epsilon})$. Consider the family of utility functions $V_\lambda^a(x) \equiv \lambda \cdot x + f^a(\|x - \lambda \frac{\lambda \cdot x}{\lambda \cdot \lambda}\|)$ defined for any $a > 0$ and $\lambda \in \Re_+^m - 0^m$. Notice that $V_\lambda^a(x) \in \mathcal{V}$ and, in our topology, $\lim_{a \mapsto \infty} V_\lambda^a(x) = \lambda \cdot x$ and $\lim_{a \mapsto 0^+} V_\lambda^a(x) = \lambda \cdot x$. Define[5] $\Omega(U_i, a) : \mathcal{U} \times \Re_+^m \mapsto\mapsto \mathcal{L}$ to be the set of linear utility functions $\lambda \cdot x$ in which the indifference curve supports the indifference curve of U_i at the allocation a. For convenience, we use the notation $\vec{\Omega}(\vec{U}, \vec{a}) \equiv \Omega(U_1, a_1) \times \Omega(U_1, a_1) \times \ldots \times \Omega(U_n, a_n)$. We now turn to two basic Lemmas.

Lemma 3 *Any $S \in \mathcal{S}^{sc}$ or any $S \in \mathcal{S}^c$ which is continuous and SP has the property that at any profile $\vec{U}$ with $U_i \in \mathcal{V}$, $S_i(\vec{U}|^i V_\lambda^a) = S_i(\vec{U})$ for each $a > 0$ and λ such that $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$.*

Proof: The following proof applies to any continuous and SP $S \in \mathcal{S}^{sc}$ or any continuous and SP $S \in \mathcal{S}^c$. Consider some profile $\vec{U}$ with $U_i \in \mathcal{V}$, and any λ such that $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$. There exists some $\bar{a}$ and a family of open neighborhoods $\mathcal{N}(a)$ of $S_i(\vec{U})$ such that, for all $a > \bar{a}$, $U_i(x) \le U_i(S_i(\vec{U})) \Rightarrow V_\lambda^a(x) < V_\lambda^a(S_i(\vec{U}))$ for all $x \in \mathcal{N}(a) - \{S_i(\vec{U})\}$. Therefore, for all $\epsilon \in [0,1]$ we must have $S_i(\vec{U}) = S_i(\vec{U}|^i \epsilon V_\lambda^a + (1-\epsilon)U_i)$. This follows because, as we vary ϵ, the allocation $S_i(\vec{U}|^i \epsilon V_\lambda^a + (1-\epsilon)U_i)$ must vary continuously but none of the nearby allocations realizable as $S_i(\vec{U}|^i W)$ (for any W in the appropriate domain $\mathcal{U}^c$ or $\mathcal{U}^{sc}$) can increase U_i (by strategy proofness) and thus must decrease V_λ^a (by construction). Thus, for each λ such that $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$ there exists some $\bar{a}$ such that, for all $a > \bar{a}$, $S_i(\vec{U}|^i V_\lambda^a) = S_i(\vec{U})$. This conclusion actually holds for any $a > 0$. To see this, assume to the contrary that there exists some $\tilde{a}$ such that $S_i(\vec{U}|^i V_\lambda^{\tilde{a}}) \neq S_i(\vec{U})$; note that $V_\lambda^{\tilde{a}}(S_i(\vec{U}|^i V_\lambda^{\tilde{a}})) > V_\lambda^{\tilde{a}}(S_i(\vec{U}))$ because S is SSP. Then, for sufficiently large a, $V_\lambda^a(S_i(\vec{U}|^i V_\lambda^{\tilde{a}})) > V_\lambda^a(S_i(\vec{U})) = V_\lambda^a(S_i(\vec{U}|^i V_\lambda^a))$ which contradicts the strategy-proofness of S. □

Lemma 4 *Any $S \in \mathcal{S}^{sc}$ or any $S \in \mathcal{S}^c$ which is continuous and SP has the property that at any profile $\vec{U}$, there exists some λ such that $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$ and $S_i(\vec{U}|^i V_\lambda^a) = S_i(\vec{U})$ for each $a > 0$.*

Proof: The following proof applies to any continuous and SP $S \in \mathcal{S}^{sc}$ or any continuous and SP $S \in \mathcal{S}^c$. Consider some profile $\vec{U}$ in the appropriate domain (either $(\mathcal{U}^c)^n$ or $(\mathcal{U}^{sc})^n$). Let U^t denote some sequence of utility functions in $\mathcal{V}$ which converge to U_i (which exists due to the density of $\mathcal{V}$ in our domain). Let λ^t be a sequence such that (1) $L_{\lambda^t} \in \Omega(U^t, S_i(\vec{U}|^i U^t))$ for each t, and (2) $\lambda^t \cdot \lambda^t = 1$. Note that, for any $a > 0$, $S_i(\vec{U}|^i U^t) = S_i(\vec{U}|^i V_{\lambda^t}^a)$. Since the λ^t's reside in a compact set, there must be at least one accumulation point; call it λ, and note that $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$ because of the fact that the U^t's converge to U_i. We can, without loss of generality, replace the sequence U^t by a subsequence such that $\lim_{t\to\infty} \lambda^t = \lambda$. Then, for any $a > 0$, $S_i(\vec{U}) = \lim_{t\to\infty} S_i(\vec{U}|^i U^t) = \lim_{t\to\infty} S_i(\vec{U}|^i V_{\lambda^t}^a) = S_i(\vec{U}|^i V_\lambda^a)$. □

Consider the following definition which is a variation of Maskin monotonicity.

Definition 7 *We say that S is* locally Maskin monotonic *(LMM) if, for any $\vec{U}$ and any set of open neighborhoods $\mathcal{N}_i$ around $S_i(\vec{U})$, we have $S(\vec{V}) = S(\vec{U})$ whenever $U_i(x) \le U_i(S_i(\vec{U})) \Rightarrow V_i(x) \le V_i(S_i(\vec{U}))$ for all allocations $x \in \mathcal{N}_i$ and all i.*

[5]The notation $\mapsto\mapsto$ denotes a correspondence.

It follows immediately that if $S \in \mathcal{S}^c$ or $\mathcal{S}^{sc}$ is locally Maskin monotonic, then S is Maskin monotonic. Conversely,

Lemma 5 *If $S \in \mathcal{S}^c$ or $\mathcal{S}^{sc}$ is continuous and Maskin monotonic, then S is locally Maskin monotonic.*

Proof: Consider some $S \in \mathcal{S}^c$ or $\mathcal{S}^{sc}$ which is continuous and Maskin monotonic. Assume, to the contrary, that S is not locally Maskin monotonic. Then, there exists some $\vec{U}$, some V_i, and some open neighborhood $\mathcal{N}_i$ of $S_i(\vec{U})$, such that (1), for all $z \in \mathcal{N}_i$, $U_i(z) \leq U_i(S_i(\vec{U})) \Rightarrow V_i(z) \leq V_i(S_i(\vec{U}))$ and (2) $S_i(\vec{U}|^iV_i) \neq S_i(\vec{U})$. Consider $S_i(\vec{U}|^iU_i + \epsilon(V_i - U_i))$ for $\epsilon \in [0,1]$. As we increase ϵ, the allocation cannot change because it must be near $S_i(\vec{U})$ because of continuity, but the SSP condition (which follows from continuity and MM) dictates that all such allocations must strictly decrease U_i and therefore cannot increase V_i. Thus, the allocation remains constant for all $\epsilon \in [0,1]$, and therefore $S_i(\vec{U}|^iV_i) = S_i(\vec{U})$. Contradiction. □

We will use the following definitions.

Definition 8 *We say that V is a* retraction *of U at x if, for all z, $U(z) \leq U(x) \Rightarrow V(z) \leq V(x)$. V is a* strict retraction *of U at x if, for all $z \neq x$, $U(z) \leq U(x) \Rightarrow V(z) < V(x)$. V is a* local retraction *of U at x if, for some open neighborhood $\mathcal{N}$ around x and all $z \in \mathcal{N}$, $U(z) \leq U(x) \Rightarrow V(z) \leq V(x)$. V is a* locally strict retraction *of U at x if, for some open neighborhood $\mathcal{N}$ around x and all $z \in \mathcal{N} - \{x\}$, $U(z) \leq U(x) \Rightarrow V(z) < V(x)$.*

If S is continuous and SP, then $S_i(\vec{U}) = S_i(\vec{U}|^iV_i)$ if V_i is a local retraction of U_i at $S_i(\vec{U})$. This follows by looking at $S_i(\vec{U}|^i\epsilon V_i + (1-\epsilon)U_i)$ as ϵ varies from 0 to 1. In what follows we will need the following result.

Lemma 6 *For every $U \in \mathcal{V}$ and every x there is a locally strict retraction $V \in \mathcal{V}$ of U at x.*

Proof: The functions V_λ^a are, for $\lambda \in \Omega(U,x)$ and sufficiently large a, locally strict retractions of U at x. □

Theorem 6 *If $S \in \mathcal{S}^c$ or $\mathcal{S}^{sc}$ is continuous and CSP, then S is locally Maskin monotonic.*

Proof: Assume that there exists some $\vec{U} \in \mathcal{V}^n$ and some $V_i \in \mathcal{U}$ such that (1) V_i is a local retraction of U_i at $S_i(\vec{U})$, and (2) $S(\vec{U}) \neq S(\vec{U}|^iV_i)$. We know that $S_i(\vec{U}) = S_i(\vec{U}|^iV_i)$, so there must be some $j \neq i$ such that $S_j(\vec{U}) \neq S_j(\vec{U}|^iV_i)$. For this j we must have $U_j(S_j(\vec{U})) = U_j(S_j(\vec{U}|^iV_i))$ or else CSP would be violated. We can assume that, without loss of generality, that V_i is a locally strict retraction of U_i at $S_i(\vec{U})$. This can be done by taking any locally strict retraction $\tilde{V}_i \in \mathcal{U}$ of U_i, which we know exists from Lemma 6 and then looking at the linear interpolation $\epsilon\tilde{V}_i + (1-\epsilon)V_i$ which is also a locally strict retraction and must, for sufficiently small ϵ, also violate LMM. Let $V_j \in \mathcal{V}$ be a locally strict retraction of U_j at $S_j(\vec{U}|^iV_i)$. Then, $S_j(\vec{U}|^{i,j}V_i, V_j) = S_j(\vec{U}|^iV_i)$ and $V_i(S_i(\vec{U}|^{i,j}V_i, V_j)) = V_i(S_i(\vec{U}|^iV_i))$. There are two cases; $S_i(\vec{U}|^{i,j}V_i, V_j) = S_i(\vec{U}|^iV_i)$ and $S_i(\vec{U}|^{i,j}V_i, V_j) \neq S_i(\vec{U}|^iV_i)$.

Assume $S_i(\vec{U}|^{i,j}V_i, V_j) = S_i(\vec{U}|^i V_i)$. Then $S_i(\vec{U}|^{i,j}V_i, V_j) = S_i(\vec{U}|^j V_j)$; this follows from Lemma 3 and because $L \in \Omega(U_i, S_i(\vec{U}|^{i,j}V_i, V_j)) \Rightarrow L \in \Omega(V_i, S_i(\vec{U}|^{i,j}V_i, V_j))$. Therefore, $V_j(S_j(\vec{U}|^j V_j)) = V_j(S_j(\vec{U}|^{i,j}V_i, V_j))$, and so either $S_j(\vec{U}|^j V_j) = S_j(\vec{U}|^{i,j}V_i, V_j)$ or $U_j(S_j(\vec{U}|^j V_j)) > U_j(S_j(\vec{U}|^{i,j}V_i, V_j)) = U_j(S_j(\vec{U}))$. The second case clearly violates SP, and the first violates SSP, since (1) $S_j(\vec{U}|^j V_j) \neq S_j(\vec{U})$, and (2) $U_j(S_j(\vec{U}|^j V_j)) = U_j(S_j(\vec{U}))$. Contradiction.

Now, assume $S_i(\vec{U}|^{i,j}V_i, V_j) \neq S_i(\vec{U}|^i V_i)$. Recall that $V_i(S_i(\vec{U}|^i V_i)) = V_i(S_i(\vec{U}|^{i,j}V_i, V_j))$. We also have the relations $U_i(S_i(\vec{U}|^{i,j}V_i, V_j)) > U_i(S_i(\vec{U}|^i V_i)) = U_i(S_i(\vec{U}))$, and $U_j(S_i(\vec{U}|^{i,j}V_i, V_j)) = U_j(S_i(\vec{U}))$. Therefore, agents i, j declaring (V_i, V_j) get a Pareto superior allocation, which contradicts CSP.

We have just shown that $S(\vec{U}) = S(\vec{U}|^i V_i)$ for any $\vec{U} \in \mathcal{V}^n$ and any $V_i \in \mathcal{U}$ which is a local retraction of U_i at $S_i(\vec{U})$. Now, consider any $\vec{U} \in \mathcal{U}^n$ and any $V_i \in \mathcal{U}$ which is a local retraction of U_i at $S_i(\vec{U})$. We can assume, without loss of generality, that $U_i(S_i(\vec{U})) = V_i(S_i(\vec{U}))$ (by considering instead the function $V_i(x) + U_i(S_i(\vec{U})) - V_i(S_i(\vec{U}))$) and $U_i(x) \geq V_i(x)$ for all $x \in \Re_+^m$ (by considering instead the function $MIN[V_i(x), U_i(x)]$ and perhaps reversing the roles of U_i and V_i). Consider some sequence $\vec{U}^t \in \mathcal{V}^n$ with $\lim_{t\to\infty} \vec{U}^t = \vec{U}$ and a sequence of V_i^t such that $\lim_{t\to\infty} V_i^t = V_i$ and each V_i^t is a local retraction of U_i^t at $S_i(\vec{U}^t)$ (such a sequence can be constructed by taking any convergent sequence V_i^t and looking at the new sequence $\tilde{V}_i^t(x) = MIN[V_i^t(x) + U_i^t(S_i(\vec{U}^t)) - V_i^t(S_i(\vec{U}^t)), U_i^t(x)]$). Then, $S(\vec{U}|^i V_i) = \lim_{t\to\infty} S(\vec{U}^t|^i V_i^t) = \lim_{t\to\infty} S(\vec{U}^t) = S(\vec{U})$. $\Box$

Thus, in all of the results that follow where the conditions are continuity, SP, and NB, we can substitute continuity and CSP, since the latter implies the former.

6.2 Results about $\mathcal{S}^{sc}$

Theorem 7 *If $S \in \mathcal{S}^{sc}$ is SSP, then S is continuous.*

Proof: Consider some $S \in \mathcal{S}^{sc}$ which is SSP. Assume, to the contrary, that there exists some $\vec{U}$ and a sequence U_i^t such that $\lim_{t\to\infty} U_i^t(x) = U_i(x)$ for all allocations x and $\lim_{t\to\infty} S_i(\vec{U}|^i U_i^t) \neq S_i(\vec{U})$. For convenience, we will introduce the notation $f(U) = S_i(\vec{U}|^i U)$ and $z = f(U_i)$, and we will assume, without loss of generality, that $U_i(f(U_i)) = 0$.

Claim: *There exists some constant C such that $f^\alpha(U) < C$ for all α and all $U \in \mathcal{U}$.*

Define $u(x) = \sum_{\alpha=1}^m x^\alpha$, and note that $U_i + u \in \mathcal{U}$. Set $C = [U_i + u](f(U_i + u)) = U_i(f(U_i + u)) + \sum_{\alpha=1}^m f^\alpha(U_i + u)$. Then, $C \geq U_i(f(U)) + u(f(U)) \geq f^\alpha(U)$, and the claim is established. Since the allocations are bounded, there must be a limit point of the infinite sequence $f(U_i^t)$. Thus, we can assume, without loss of generality (by picking a subsequence of the original sequence) that there is some point y such that $\lim_{t\to\infty} f(U_i^t) = y$.

Claim: $\lim_{t\to\infty} U_i(f(U_i^t)) = U_i(f(U_i))$

Since U_i is continuous, $\lim_{t\to\infty} U_i(f(U_i^t)) = U_i(\lim_{t\to\infty} f(U_i^t)) = U_i(y)$. Clearly $U_i(f(U_i^t)) \leq U_i(f(U_i))$ for all t. Thus, if above claim is false, we can choose a new sequence U_i^t (which is a subsequence of the original sequence) and some $\epsilon > 0$ such that $U_i(f(U_i^t)) \leq U_i(f(U_i)) - \epsilon$.

For any compact neighborhood $\mathcal{N}$ of y (with y in the interior of $\mathcal{N}$ with respect to $\Re^m_+$), there exists some T such that, for all $t > T$, we have (1) $\|U_i^t(x) - U_i(x)\| < \frac{\epsilon}{3}$ for all $x \in \mathcal{N}$, and (2) $f(U_i^t) \in \mathcal{N}$. Thus, for all $t > T$, we have $0 \geq U_i^t(f(U_i)) - U_i^t(f(U_i^t)) = U_i(f(U_i)) + [U_i^t(f(U_i)) - U_i(f(U_i))] - U_i(f(U_i^t)) - [U_i^t(f(U_i^t)) - U_i(f(U_i^t))] = [U_i(f(U_i)) - U_i(f(U_i^t))] + [U_i^t(f(U_i)) - U_i(f(U_i))] - [U_i^t(f(U_i^t)) - U_i(f(U_i^t))] \geq \frac{\epsilon}{3} > 0$. Contradiction.

Thus, $U_i(y) = U_i(z)$ (recall that $z \equiv f(U_i)$), and we can assume, without loss of generality (by picking a subsequence of the original sequence) that $U_i(f(U_i^t)) > U_i(f(U_i^{t-1}))$ for all $t > 1$.

Claim: *There is no $W \in \mathcal{U}$ such that $W(y) > W(z)$.*

Assume, to the contrary, that there is some W such that $W(y) > W(z)$. Consider $V(x) = MIN[W(x) - W(y), U_i(x) - U_i(y)]$; clearly $U_i(x) \leq U_i(y) \Rightarrow V(x) \leq V(y)$, $U_i(x) < U_i(y) \Rightarrow V(x) < V(y)$, and $V(y) > V(z)$. The SSP condition means that $f(U) \neq z \Rightarrow U_i(f(U)) < U_i(z)$. Consequently, $V(f(V)) < V(y)$. However, since $\lim_{t\to\infty} V(f(U_i^t)) = V(y)$, so there is some T such that for all $t > T$ we have $V(f(U_i^t)) > V(f(V))$. Contradiction, thereby establishing the claim.

In $\mathcal{U}^{sc}$, whenever two points lie on an indifference curve for each of the points there is always some utility that prefers one point over the other. Therefore, we have reached a contradiction so the Theorem must hold. □

This Theorem, along with Theorem 5, implies that in $\mathcal{U}^{sc}$, S is SSP if and only if S is continuous and SP. It is an open question as to whether a similar result holds for $\mathcal{U}^c$.

Theorem 8 *If $S \in \mathcal{S}^{sc}$ is continuous, SP, and NB, then S is SCSP.*

Proof: We know, from Lemma 3, that S must be MM, CNB, and WCSP. Assume, to the contrary, that there is some $\vec{U} \in (\mathcal{U}^{sc})^n$ and $\vec{V} \in (\mathcal{U}^{sc})^n$ with $S(\vec{U}) \neq S(\vec{V})$ such that $U_i(S_i(\vec{V})) \geq U_i(S_i(\vec{U}))$ for each $i \in E$, where $E = \{i | U_i \neq V_i\}$. Define $\tilde{E} = \{i \in E | S_i(\vec{V}) \neq S_i(\vec{U})\}$; from the CNB condition we know that this set is nonempty. For each $i \in E - \tilde{E}$, define $W_i(x) = MIN[U_i(x) - U_i(S_i(\vec{U})), V_i(x) - V_i(S_i(\vec{U}))]$. Note that for each such i, $S(\vec{U}) = S(\vec{U}|^i W_i)$ and $S(\vec{V}) = S(\vec{V}|^i W_i)$. Thus, without loss of generality we can choose $\vec{U}$ and $\vec{V}$ so that $E = \tilde{E}$; we will assume we have done so. For each $i \in E$, we know that there exists some λ such that (1) $L_\lambda(x) \in \Omega(U_i, S_i(\vec{U}))$ and (2) $S_i(\vec{U}|^i V_\lambda^a) = S_i(\vec{U})$ for each $a > 0$. The strict convexity of U_i requires that, for sufficiently large a, $V_\lambda^a(S_i(\vec{U})) < V_\lambda^a(S_i(\vec{V}))$ for each $i \in E$. Set $W_i = V_\lambda^a$ for each such i. For all $i \notin E$, set $W_i = U_i$. Then, $S(\vec{U}) = S(\vec{W})$, so $W_i(S_i(\vec{V})) > W_i(S_i(\vec{W}))$ for each i such that $W_i \neq V_i$. This violates the WCSP property of S. □

The obvious corollary to this result is that Nash implementability plus continuity imply SCSP.

Corollary 4 *If $S \in \mathcal{S}^{sc}$ is continuous and Nash implementable, then S is SCSP.*

Proof: This follows directly from Theorem 8 and the fact that NI implies SP + NB. □

6.3 Results about $\mathcal{S}^c$

Note that the above results required the strict convexity of utilities. Here, we relax that condition.

Lemma 7 *If $S \in \mathcal{S}^c$ is continuous and SP, then at any profile $\vec{U}$ such that $U_i \in \mathcal{V}$, $S_i(\vec{U}) = S_i(\vec{U}|^i L_i)$ for all $L_i \in \Omega(U_i, S_i(\vec{U}))$.*

Proof: This follows directly from taking the limit $a \to \infty$ in Lemma 3. $\square$

Lemma 8 *If $S \in \mathcal{S}^c$ is continuous and SP, then there exists some $L_i \in \Omega(U_i, S_i(\vec{U}))$ such that $S_i(\vec{U}) = S_i(\vec{U}|^i L_i)$.*

Proof: This follows directly from taking the limit $a \to \infty$ in Lemma 4. $\square$

Theorem 9 *If $S \in \mathcal{S}^c$ is continuous, SP, and NB, then for each $\vec{U} \in (\mathcal{U}^c)^n$, there exists some $\vec{L} \in \vec{\Omega}(\vec{U}, S(\vec{U}))$ such that $S(\vec{U}) = S(\vec{L})$.*

Proof: First, by Lemma 8, there exists some $L_1 \in \Omega(U_1, S_1(\vec{U}))$ such that $S_1(\vec{U}|^1 L_1) = S_1(\vec{U})$. Because of nonbossiness, we have $S(\vec{U}|^1 L_1) = S(\vec{U})$. Applying this reasoning now to agent 2 and the profile $(\vec{U}|^1 L_1)$, and so on, we find some $\vec{L} \in \vec{\Omega}(\vec{U}, S(\vec{U}))$ such that $S(\vec{U}) = S(\vec{L})$. $\square$

Theorem 10 *If $S, \tilde{S} \in \mathcal{S}^c$ are continuous, SP, and NB, and have the property that $S(\vec{L}) = \tilde{S}(\vec{L})$ for each $\vec{L} \in \mathcal{L}^n$, then $S = \tilde{S}$.*

Proof: Consider any $\vec{U} \in \mathcal{V}^n$. Since, by Lemma 7 and the nonbossiness condition, we can conclude that $S(\vec{U}) = S(\vec{L})$ for any $\vec{L} \in \vec{\Omega}(\vec{U}, S(\vec{U}))$. Furthermore, by assumption, we know $S(\vec{L}) = \tilde{S}(\vec{L})$ for each such $\vec{L} \in \vec{\Omega}(\vec{U}, S(\vec{U}))$. Since $\tilde{S}$ is MM, and each U_i is a retraction of each $L_i \in \Omega(U_i, S_i(\vec{U}))$ at $S_i(\vec{U})$, we know that $\tilde{S}(\vec{U}) = \tilde{S}(\vec{L})$. Therefore, $S(\vec{U}) = \tilde{S}(\vec{U})$ for all $\vec{U} \in \mathcal{V}^n$. Since S and $\tilde{S}$ are continuous, and $\mathcal{V}$ is dense in $\mathcal{U}^c$, we have $S(\vec{U}) = \tilde{S}(\vec{U})$ for all $\vec{U} \in \mathcal{U}^{cn}$. $\square$

Thus, in the domain $\mathcal{S}^c$, social choice functions are completely characterized by their behavior on the linear profiles.

7 References

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